Partial Differential Equations

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[1]These lecture notes are a draft and likely to contain mistakes. Please report any typos, errors, or suggestions to jonas.lampart@u-bourgogne.fr. Version of February 9, 2026

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1. Introduction

A partial differential equation (PDE) is an equation whose 'unknown' is a function u, and in which (partial) derivatives of that function appear. This is similar to an ordinary differential equation (ODE) but the difference is that the unknown function

$$u : \mathbb{R}^d \to \mathbb{R} \text{ (or } \mathbb{C})$$

depends on more than one variable, $d \geq 2$, and derivatives in different directions play a role. Such equations, or systems of equations, arise in many contexts mathematics and applications in physics, engineering, and the sciences – such as electrodynamics, quantum mechanics, dynamics of weather and climate, and the description of materials.

1.1. Examples

1. The heat equation
$$\partial_t u(t, x) = \Delta_x u(t, x) \tag{1.1}$$
describes diffusion of heat in a (homogeneous, isotropic) medium.

2. Schrödinger's equation
$$\mathrm{i}\partial_t \psi(t, x) = -\Delta_x \psi(t, x) + V(x)\psi(t, x) \tag{1.2}$$
describes the wave-function of a quantum particle in an external potential V.

3. The Poisson equation
$$\Delta u(x) = \rho(x) \tag{1.3}$$
gives the electric potential generated by the (static) charge distribution ρ. Maxwell's equations give a more complete description of electrodynamics.

4. The Euler equation
$$\begin{cases} \partial_t v(t, x) + v(t, x) \cdot D_x v(t, x) + \operatorname{grad}_x p(t, x) = 0 \\ \operatorname{div}_x v(t, x) = 0 \end{cases} \tag{1.4}$$
describes the velocity field $v : \mathbb{R}^d \to \mathbb{R}^d$ and pressure $p : \mathbb{R}^d \to \mathbb{R}$ of an incompressible, inviscid fluid. Similar systems, like the Navier-Stokes equations, are used to model the dynamics of fluids and gases with different properties, e.g. water waves or atmospheric currents.

5. The Cauchy-Riemann equations

$$\begin{cases} \partial_x u(x,y) - \partial_y v(x,y) = 0 \\ \partial_y u(x,y) + \partial_x v(x,y) = 0 \end{cases} \tag{1.5}$$

are satisfied by the real and imaginary part of every holomorphic function $f = u + \mathrm{i}v : \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{C}$.

Let $\alpha \in \mathbb{N}_0^d$ be a 'multi-index' and set

$$\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}}, \tag{1.6}$$

where $|\alpha| = \sum_{j=1}^d \alpha_j$. That is, α_j is the number of partial derivatives in direction j and $|\alpha|$ is the total number of derivatives. Since for $u \in C^k(U, \mathbb{C}^n)$ the partial derivatives can be taken in any order, we can thus express the tensor $D^k u$ by

$$(D^k u)_{j_1,\dots,j_k} = \frac{\partial^k u}{\partial x_{j_k} \cdots \partial x_{j_1}} = \partial^\alpha u \tag{1.7}$$

where α_i is the number of partial derivatives taken in the i-th direction, and $|\alpha| = k$.

Note that we have the generalised Leibniz rule

$$\partial^\alpha (fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g), \tag{1.8}$$

where $\beta \leq \alpha$ if $\beta \leq \alpha$ if $\beta_j \leq \alpha_j$ for all $j = 1, \dots, d$, and the binomial coefficients are generalised as

$$\binom{\alpha}{\beta} = \prod_{j=1}^d \binom{\alpha_j}{\beta_j}. \tag{1.9}$$

Definition 1.1 (Linear PDE)**.** A PDE is called (inhomogneous) linear PDE of order k if it has the form

$$\sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha u = f(x), \tag{1.10}$$

where $a_\alpha : \mathbb{R}^d \to \mathbb{C}^{n\times n}$, for $|\alpha| \leq k$, and $f : \mathbb{R}^d \to \mathbb{C}^n$. The functions a_α are called the coefficients, and the PDE is called homogeneous if $f = 0$.

Question 1.2. Which of the examples in Sect. 1.1 are linear (in-) homogeneous PDEs?

2. Linear PDEs with constant coefficients and the Fourier transform

A particularly simple case of linear differential equations are those with constant coefficients, where the functions $a_\alpha(x) \equiv a_\alpha$ are independent of x. These can be transformed into simpler equations by the Fourier transform.

For $f \in L^1(\mathbb{R}^d)$, the Fourier transform is defined by

$$\hat{f}(p) = (\mathscr{F} f)(p) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}p\cdot x} f(x) \mathrm{d}x. \tag{2.1}$$

Formally, we have with $p^\alpha = \prod_{j=1}^d p_j^{\alpha_j}$

$$\begin{aligned} p^\alpha \hat{f}(p) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} p^\alpha \mathrm{e}^{-\mathrm{i}p\cdot x} f(x) \mathrm{d}x \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (-\mathrm{i})^{-|\alpha|} (\partial_x^\alpha \mathrm{e}^{-\mathrm{i}p\cdot x}) f(x) \mathrm{d}x \\ &\overset{!}{=} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (\mathrm{i})^{|\alpha|} (-1)^{|\alpha|} \mathrm{e}^{-\mathrm{i}p\cdot x} \partial_x^\alpha f(x) \mathrm{d}x \\ &= (-\mathrm{i})^{|\alpha|} \widehat{\partial_x^\alpha f}(p), \end{aligned}$$

but the integration by parts (without boundary terms!) in the penultimate step certainly needs justification.

If we accept this identity, the linear PDE of Def. 1.1 becomes after transformation

$$\Big(\sum_{|\alpha| \le k} a_\alpha (\mathrm{i}p)^\alpha \Big) \hat{u}(p) = \hat{f}(p). \tag{2.2}$$

Any solution then satisfies, formally,

$$\hat{u}(p) \overset{!}{=} \Big(\sum_{|\alpha| \le k} a_\alpha (\mathrm{i}p)^\alpha \Big)^{-1} \hat{f}(p).$$

To recover the solution u, however, we will need to invert the Fourier transform.

2.1. Basic properties

A few important properties of the Fourier transform of $f \in L^1(\mathbb{R}^d)$ follow immediately from the definition.

Proposition 2.1. *Let $f \in L^1(\mathbb{R}^d)$, denote by $\hat{f}$ its Fourier transform* (2.1) *and denote by $Rf(x) = f(-x)$ the reflection at $x = 0$ and $T_a f(x) = f(x-a)$ the translation. Then*

a) $\widehat{T_a f} = \mathrm{e}^{-\mathrm{i}ap} \hat{f}$

b) $T_a \hat{f} = \widehat{\mathrm{e}^{\mathrm{i}ax} f}$

c) $R\hat{f} = \widehat{Rf}$

d) $\overline{\hat{f}} = R\hat{\overline{f}}$

e) If f is real and even ($Rf = f$) then $\hat{f}$ is also real and even.

Proof. Properties *a)–d)* follow from simple changes of variables (exercise). Property *e)* follows by combining *c*) and *d*). □

The Dominated Convergence Theorem A.9 also yields that $\hat{f}$ is continuous.

Lemma 2.2. *Let $f \in L^1(\mathbb{R}^d)$ and $\hat{f}$ its Fourier transform* (2.1)*, then $\hat{f}$ is continuous.*

Proof. Let $p_n \to p$ be a convergent sequence. Then since $|\mathrm{e}^{-\mathrm{i}p_n x} f(x)| \leq |f(x)| \in L^1(\mathbb{R}^d)$

$$\lim_{n\to\infty} \hat{f}(p_n) = \lim_{n\to\infty} \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{-\mathrm{i}p_n x} f(x) \mathrm{d}x = \frac{1}{(2\pi)^{d/2}} \int \lim_{n\to\infty} \mathrm{e}^{-\mathrm{i}p_n x} f(x) \mathrm{d}x = \hat{f}(p) \tag{2.3}$$

by Dominated Convergence A.9, which proves the claim. □

2.2. The Schwartz space $\mathscr{S}$

In order to make the formal calculations from the introduction rigorous and derive consequences for the solutions to the PDE, we start by introducing a class of functions on which the calculations can easily be justified. We will later expand beyond this class by approximation arguments.

A good framework to consider identities such as (2.2) is the space of Schwartz functions, where we can

- differentiate
- multiply by polynomials
- define the Fourier transform and its inverse.

Definition 2.3. The Schwartz space is

$$\mathscr{S}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) \middle| \forall \alpha, \beta \in \mathbb{N}_0^d : \sup_{x\in\mathbb{R}^d} |x^\alpha \partial_x^\beta f(x)| < \infty \right\}. \tag{2.4}$$

A sequence f_n, $n \in \mathbb{N}$ in $\mathscr{S}$ converges to $f \in \mathscr{S}$ iff

$$\forall \alpha, \beta \in \mathbb{N}_0^d : \lim_{n\to\infty} \|f_n - f\|_{\alpha,\beta} = 0, \tag{2.5}$$

where

$$\|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} |x^\alpha \partial_x^\beta f(x)|. \tag{2.6}$$

A map $T : \mathscr{S}(\mathbb{R}^d) \to X$ into a metric space X is continuous iff T is sequentially continuous, that is, if for every sequence f_n converging to $f \in \mathscr{S}(\mathbb{R}^d)$

$$\lim_{n\to\infty} Tf_n = Tf \tag{2.7}$$

converges in X.

Question 2.4. Which of the following functions are elements of $\mathscr{S}(\mathbb{R})$?

1. $x \mapsto \cos(x)$,
2. $x \mapsto \cosh(x)^{-1} = 2(e^x + e^{-x})^{-1}$,
3. $x \mapsto e^{-|x|}$,
4. $x \mapsto e^{-x^2}$.

Remark 2.5. The space $\mathscr{S}$ is a complete metric space with the distance

$$d(f,g) = \sum_{n\in\mathbb{N}_0} 2^{-n} \max_{|\alpha|+|\beta|=n} \frac{\|f-g\|_{\alpha,\beta}}{1+\|f-g\|_{\alpha,\beta}}. \tag{2.8}$$

The notion of convergence defined above is the same as the convergence in the metric d.

Remark 2.6. Functions in $\mathscr{S}$ are smooth by definition, and decrease faster than any inverse polynomial. Hence $\mathscr{S} \subset L^\infty$ with $\|f\|_\infty = \|f\|_{0,0}$, and $\mathscr{S} \subset L^p$ for any $1 \le p < \infty$, as by the multinomial formula

$$\begin{aligned}
|f(x)| &\le (1+(x^2)^d)^{-1} \sup_{y\in\mathbb{R}^d} |(1+(y^2)^d) f(y)| \\
&\le (1+x^{2d})^{-1} \sup_{y\in\mathbb{R}^d} \Big|\Big(1 + \sum_{|\alpha|=d} \frac{d!}{\alpha!} y^{2\alpha}\Big) f(y)\Big| \\
&\le (1+x^{2d})^{-1} (\|f\|_{0,0} + \sum_{|\alpha|=d} \frac{d!}{\alpha!} \|f\|_{2\alpha,0}),
\end{aligned}$$

and

$$\int (1+x^{2d})^{-p} < \infty \tag{2.9}$$

for $p \ge 1$.

For $f, g \in \mathscr{S}(\mathbb{R}^d)$ we define the convolution by

$$(f * g)(x) := \int f(x-y) g(y) \mathrm{d}y. \tag{2.10}$$

Lemma 2.7. *Let $f, g \in \mathscr{S}(\mathbb{R}^d)$ with $\int g = 1$ and set $g_n(x) = n^d g(nx)$, then*

$$\lim_{n\to\infty} (f * g_n)(x) = f(x).$$

Proof. By a change of variable

$$\int f(x-y) n^d g(ny) \mathrm{d}y = \int f(x - n^{-1}y) g(y) \mathrm{d}y. \tag{2.11}$$

Now the integrand converges pointwise to $f(x)g(y)$ and is bounded by $\|f\|_\infty |g(y)| \in L^1$, so the integral converges to $f(x) \int g = f(x)$ by Dominated Convergence. □

With this Lemma, we can prove the Fourier inversion theorem on $\mathscr{S}$.

Proposition 2.8. *Define*

$$(\mathscr{F}^{-1} f)(x) := \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{\mathrm{i}px} f(p) \mathrm{d}p.$$

Then for all $f \in \mathscr{S}(\mathbb{R}^d)$,

$$f = \mathscr{F}^{-1}\mathscr{F} f = \mathscr{F}\mathscr{F}^{-1} f.$$

Proof. We admit that $\hat{f} \in \mathscr{S}$, which is proved in Proposition 2.9 below. Let $g(x) = \mathrm{e}^{-x^2/2}$ and $g_n(x) = g(n^{-1}x)$. Then

$$(\mathscr{F}^{-1}\hat{f})(x) = \lim_{n\to\infty} \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{\mathrm{i}px} g_n(p) f(p) \mathrm{d}p \tag{2.12}$$

by Dominated Convergence. On the other hand, by Fubini,

$$\frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{\mathrm{i}px} g_n(p) \hat{f}(p) \mathrm{d}p = \frac{1}{(2\pi)^d} \int \mathrm{e}^{\mathrm{i}px} \mathrm{e}^{-\mathrm{i}py} g_n(p) f(y) \mathrm{d}y \mathrm{d}p = \frac{(\hat{g}_n * f)(x)}{(2\pi)^{d/2}}. \tag{2.13}$$

Now $\hat{g}_n(x) = n^d \mathrm{e}^{-n^2 x^2/2}$ (see Problem 2), so by the preceding lemma

$$\lim_{n\to\infty} (\hat{g}_n * f)(x) = f(x) \int g = (2\pi)^{d/2} f(x), \tag{2.14}$$

and thus $(\mathscr{F}^{-1}\hat{f})(x) = f(x)$. The proof for $\mathscr{F}(\mathscr{F}^{-1} f)(x) = f(x)$ is the same. □

Proposition 2.9. *The Fourier transform $\mathscr{F}$ is a linear and continuous map*

$$\mathscr{F} : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d), \qquad f \mapsto \hat{f}.$$

Its continuous inverse is given by $\mathscr{F}^{-1}$. Moreover, the identities

$$(\partial^\alpha \mathscr{F} f)(p) = (\mathscr{F}(-\mathrm{i}x)^\alpha f)(p) \tag{2.15}$$

$$p^\alpha (\mathscr{F} f)(p) = \mathscr{F}\big((-\mathrm{i})^{|\alpha|} \partial^\alpha) f\big)(p) \tag{2.16}$$

hold for all $f \in \mathscr{S}(\mathbb{R}^d)$, $\alpha \in \mathbb{N}_0^d$.

Proof. We prove that $\hat{f}$ is smooth and the first identity by induction on $|\alpha|$. For $|\alpha| = 0$ we only need to prove that $\hat{f}$ is continuous, which is Lemma 2.2.

Now assume the statement holds for all $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq k$ and let $|\alpha| = k+1$. Then there are $\beta \in \mathbb{N}_0^d$ and $j \in \{1, \dots, d\}$ with $\alpha = \beta + e_j$. Denote $g = \partial^\beta \hat{f}$. By the theorem on parameter-dependent integrals A.10 and the induction hypothesis

$$g(p) = \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{-\mathrm{i}px} (-\mathrm{i}x)^\beta f(x) \mathrm{d}x \tag{2.17}$$

is differentiable, with

$$\partial_{p_j} g(p) = \frac{1}{(2\pi)^{d/2}} \int \partial_{p_j} \mathrm{e}^{-\mathrm{i}px} (-\mathrm{i}x)^\beta f(x) \mathrm{d}x = \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{-\mathrm{i}px} (-\mathrm{i}x)^\alpha f(x) \mathrm{d}x. \tag{2.18}$$

This completes the induction.

For the second identity, we use that

$$\begin{aligned} p_j \hat{f}(p) &= \frac{1}{(2\pi)^{d/2}} \int p_j \mathrm{e}^{-\mathrm{i}px} f(x) \mathrm{d}x = \frac{1}{(2\pi)^{d/2}} \int \mathrm{i}\partial_{x_j} \mathrm{e}^{-\mathrm{i}px} f(x) \mathrm{d}x \\ &= \frac{1}{(2\pi)^{d/2}} \int \mathrm{i}\partial_{x_j} (\mathrm{e}^{-\mathrm{i}px} f(x)) - \mathrm{i}\mathrm{e}^{-\mathrm{i}px} \partial_{x_j} f(x) \mathrm{d}x. \end{aligned} \tag{2.19}$$

The integral of the derivative vanishes, because for $h \in \mathscr{S}$ by Fubini's Theorem A.6 and the fundamental theorem of calculus

$$\begin{aligned} \int \partial_{x_j} h(x) \mathrm{d}x &= \int_{|x_j| \leq R} \partial_{x_j} h(x) \mathrm{d}x + \int_{|x_j| > R} \partial_{x_j} h(x) \mathrm{d}x \\ &= \int_{\mathbb{R}^{d-1}} h(x) \Big|_{x_j = -R}^{x_j = R} + \int_{|x_j| > R} \partial_{x_j} h(x) \mathrm{d}x, \end{aligned} \tag{2.20}$$

and

$$\lim_{R \to \infty} \int_{\mathbb{R}^{d-1}} h(x) \Big|_{x_j = -R}^{x_j = R} = 0 = \int_{|x_j| > R} \partial_{x_j} h(x) \mathrm{d}x, \tag{2.21}$$

since h vanishes faster than any polynomial. This proves the second identity in the case $|\alpha| = 1$, from which the general case follows by induction, like the first.

We have shown that $\hat{f}$ is smooth, so to show that $\hat{f} \in \mathscr{S}$ we need to show that $\|\hat{f}\|_{\alpha,\beta}$ is finite. Using the identities, we find using the Leibniz rule (see Problem 4)

$$\begin{aligned} \|\hat{f}\|_{\alpha,\beta} &= \sup_{p \in \mathbb{R}^d} \left| (\mathscr{F} \partial^\alpha x^\beta)(p) f \right| \\ &\leq \frac{1}{(2\pi)^{d/2}} \int (1 + x^{2d})^{-1} \sup_{y \in \mathbb{R}^d} (1 + y^{2d}) |\partial^\alpha y^\beta f(y)| \\ &\leq C \sum_{\substack{|\gamma| \leq |\beta| + 2d \\ |\delta| \leq |\alpha|}} \|f\|_{\gamma,\delta} \end{aligned} \tag{2.22}$$

for some constant C, and hence $\hat{f} \in \mathscr{S}$. Moreover, $\mathscr{F}$ is continuous in $f = 0$ by the bound (2.22), so it is continuous by linearity. Continuity of the inverse follows from $\mathscr{F}^{-1} = R\mathscr{F}$. □

Corollary 2.10. *Let $f, g \in \mathscr{S}(\mathbb{R}^d)$, then*

$$\int f(x)\hat{g}(x) = \int \hat{f}(x)g(x),$$

and

$$\int |\hat{f}|^2(p)\mathrm{d}p = \int |f|^2(x)\mathrm{d}x.$$

Proof. The first statement follows directly from Fubini's theorem. The second is a consequence of this and the Fourier inversion formula together with $\mathscr{F}^{-1}f(x) = \hat{f}(-x) = Rf$ and Proposition 2.1d), i.e.,

$$\int \hat{f}(p)\overline{\hat{f}}(p)\mathrm{d}p = \int f(x)\hat{\overline{\hat{f}}}(x)\mathrm{d}x = \int f(x)\widehat{R\hat{\overline{f}}}(x)\mathrm{d}x = \int |f(x)|^2\mathrm{d}x. \tag{2.23}$$

□

Example 2.11. Let $z \in \mathbb{C}$, $f \in \mathscr{S}(\mathbb{R}^d)$ and consider the linear PDE

$$(\Delta + z)u = f. \tag{2.24}$$

Assuming that $u \in \mathscr{S}$, we can take the Fourier transform and obtain

$$(-p^2 + z)\hat{u}(p) = \hat{f}(p). \tag{2.25}$$

If $z \in \mathbb{C} \setminus \mathbb{R}_+$, then $-p^2 + z \neq 0$, and

$$\hat{u}(p) = (-p^2 + z)^{-1}\hat{f}(p) \in \mathscr{S}. \tag{2.26}$$

In this case, the unique solution $u \in \mathscr{S}(\mathbb{R}^d)$ to (2.24) is given by

$$u(x) = \mathscr{F}^{-1}(-p^2 + z)^{-1}\hat{f}. \tag{2.27}$$

Uniqueness holds only with the requirement that $u \in \mathscr{S}$. Without this hypothesis, we can add any solution v of the homogeneous equation

$$(\Delta + z)v = 0, \tag{2.28}$$

for example $v_\pm = \mathrm{e}^{\pm\sqrt{-z}x}$ for $d = 1$, $z \neq 0$. Note that these solutions are not elements of $\mathscr{S}$, as they do not decay for $|x| \to \infty$!

If $z \in \mathbb{R}_+$ the situation is more complicated as $-p^2 + z$ is not smoothly invertible, but if $\hat{f}$ has the same zeros the solution might still be an element of $\mathscr{S}$.

Example 2.12. (The heat equation on $\mathscr{S}$) If we take the Fourier transform of the heat equation

$$\partial_t u = \Delta u \tag{2.29}$$

in both t and x, we obtain

$$(\mathrm{i}\tau + p^2)\mathscr{F}_{t,x}u = 0. \tag{2.30}$$

In the best case this would tell us that $u = 0$ (though this is not clear since the multiplier vanishes at $(\tau, p) = 0$). However, the equation is an evolution equation and $\mathscr{S}(\mathbb{R} \times \mathbb{R}^d)$ is not a natural space for the solutions. Indeed, $u \in \mathscr{S}(\mathbb{R} \times \mathbb{R}^d)$ would mean that $u(t, x) \to 0$ for $t \to \pm\infty$, but instead of this restriction we should rather specify initial data, as for ODEs.

If we only take the Fourier transform in x, we obtain

$$\partial_t \hat{u}(t, p) = -p^2 \hat{u}(t, p). \tag{2.31}$$

If we fix an initial condition $\hat{u}_0(p) = \hat{u}(0, p) \in \mathscr{S}(\mathbb{R}^d)$ the equation is an ODE initial value problem for every p. The unique solution is

$$\hat{u}(t, p) = \mathrm{e}^{-p^2 t} \hat{u}_0(p), \tag{2.32}$$

and for every $t \geq 0$ this is again an element of $\mathscr{S}(\mathbb{R}^d)$. Moreover, $\lim_{t \to 0} \mathrm{e}^{-p^2 t} \hat{u}_0(p) = u_0$ in $\mathscr{S}(\mathbb{R}^d)$.

With this we can see that there exists a unique function

$$(t, x) \mapsto u(t, x), \qquad u \in C^1((0, \infty) \times \mathbb{R}^d, \mathbb{C}), \qquad u(t, \cdot) \in \mathscr{S}(\mathbb{R}^d) \tag{2.33}$$

satisfying the heat equation (2.29) and such that

$$\lim_{t \to 0} u(t, \cdot) = u_0 \tag{2.34}$$

in $\mathscr{S}(\mathbb{R}^d)$.

2.3. Convolution and approximation

The convolution of functions appears frequently in formulas for solutions of PDEs. It is also an important technical tool that enables us to approximate functions with singularities by smooth functions.

This will allow us to extend the maps we have defined on $\mathscr{S}$ to L^p by continuity, using the following:

Theorem 2.13 (B.L.T. Theorem)**.** *Let X, Y be Banach spaces and $D \subset X$ a dense subspace. Suppose $A : D \to Y$ is a bounded linear transformation, then there exists a unique bounded linear transformation $\overline{A} : X \to Y$ that extends A, and $\|\overline{A}\| = \|A\|$ holds.*

Proof. By [FA, Prop.1.7.2], A is continuous so the idea is to extend in such a way that preserves continuity.

Since $\overline{D} = X$, every $x \in X \setminus D$ is a limit point of of D, i.e., there exist $x_n \in D$, $n \in \mathbb{N}$, so that $x_n \to x$ as $n \to \infty$. The sequence x_n is Cauchy in X, and because A is bounded, we have

$$\|Ax_n - Ax_m\|_Y \leq \|A\| \|x_n - x_m\|_X, \tag{2.35}$$

so the sequence Ax_n is Cauchy in Y. Since Y is complete, it thus converges to a limit $y \in Y$. We set

$$\overline{A}x := y. \tag{2.36}$$

This is well defined, for if $\tilde{x}_n \to x$ is another sequence, then $\tilde{x}_n - x_n \to 0$ and thus

$$\lim_{n\to\infty} A\tilde{x}_n = y + \lim_{n\to\infty} A(\tilde{x}_n - x_n) = y, \tag{2.37}$$

by continuity of A. Linearity of $\overline{A}$ follows from linearity of A and the limit. This extension is unique, for if $\tilde{A}$ were another bounded extension, it would be continuous by [FA, Prop.1.7.2] and $\tilde{A}x = y = \overline{A}x$ follows.

Moreover, we have by continuity of the norm

$$\|\overline{A}x\|_Y = \|\lim_{n\to\infty} Ax_n\|_Y = \lim_{n\to\infty} \|Ax_n\|_Y \leq \|A\| \lim_{n\to\infty} \|x_n\|_X = \|A\|\|x\|_X, \tag{2.38}$$

so $\|\overline{A}\| \leq \|A\|$. We also have $\|A\| \leq \|\overline{A}\|$, since in one case the supremum is over D and in the other over X, which is larger. Thus $\overline{A}$ is bounded with $\|\overline{A}\| = \|A\|$. □

The convolution of f with $g \in \mathscr{S}$ has a smoothing effect, which we will use this to approximate arbitrary elements of $L^p(\mathbb{R}^d)$ by smooth functions.

Theorem 2.14. *Let $1 \leq p \leq \infty$, $f \in L^p(\mathbb{R}^d)$ and $g \in \mathscr{S}(\mathbb{R}^d)$. Then $f * g$ is well defined, and*

a) $f * g \in C^\infty(\mathbb{R}^d)$;

b) If $(1 + x^2)^r f(x) \in L^p(\mathbb{R}^d)$ for some $r > 0$, then $(1 + x^2)^r (f * g)(x) \in L^\infty(\mathbb{R}^d)$;

c) If $f \in \mathscr{S}(\mathbb{R}^d)$ then

$$\widehat{f * g} = (2\pi)^{d/2} \hat{f}\hat{g}, \qquad \widehat{fg} = (2\pi)^{-d/2} \hat{f} * \hat{g}.$$

Proof. Since $g \in \mathscr{S}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ with $p^{-1} + q^{-1} = 1$, we have $y \mapsto f(x - y)g(x) \in L^1(\mathbb{R}^d)$ by the Hölder inequality, so the integral is well defined.
a): Changing variables with $z = x - y$ we have

$$(f * g)(x) = \int f(z)g(x - z)\mathrm{d}z. \tag{2.39}$$

For given x_0, we consider a neighbourhood $U = B(x_0, 1)$. Since $g \in \mathscr{S}$, the derivatives of the integrand satisfy for $x \in U$

$$|\partial_x^\alpha f(z)g(x - z)| \leq C|f|(z)(1 + |x_0 - z|^{2d})^{-1} \tag{2.40}$$

with some $C > 0$. The right hand side is integrable, since $(1 + |x_0 - z|^{2d})^{-1} \in L^q(\mathbb{R}^d)$ for all $1 \leq q \leq \infty$. Thus, $f * g$ is smooth by Theorem A.10

b): We write

$$(1 + x^2)^r (f * g)(x) = \int (1 + |x - y|^2)^r f(x - y)(1 + y^2)^r g(y) \Big(\frac{1 + x^2}{(1 + |x - y|^2)(1 + y^2)} \Big)^r \mathrm{d}y. \tag{2.41}$$

Now

$$\frac{1+x^2}{(1+|x-y|^2)(1+y^2)} \le 4 \tag{2.42}$$

since for $|x-y| \le |x|/2$ we have $|y| \ge |x|/2$, and thus

$$\frac{1+x^2}{(1+|x-y|^2)(1+y^2)} \le \frac{1+x^2}{1+x^2/4} \le 4, \tag{2.43}$$

and the same argument with reversed roles in the denominator for $|x-y| \ge |x|/2$. Thus, by Hölder's inequality (A.7),

$$\begin{aligned} &\left| \int (1+|x-y|^2)^r f(x-y)(1+y^2)^r g(y) \Big(\frac{1+x^2}{(1+|x-y|^2)(1+y^2)} \Big)^r \right| \\ &\quad \le 4^r \Big(\int |(1+|x-y|^2)^r f(x-y)|^p \mathrm{d}y \Big)^{1/p} \Big(\int (1+y^2)^r |g(y)|^q (y) \mathrm{d}y \Big)^{1/q}, \end{aligned} \tag{2.44}$$

which proves the claim by a change of variables in the integral involving f.

c) By *b)*, $f * g \in L^1(\mathbb{R}^d)$, so the Fourier transform is well defined. Now

$$\widehat{f * g} = \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{-\mathrm{i}p(x-y)-\mathrm{i}py} f(x-y)g(y)\mathrm{d}y\mathrm{d}x = (2\pi)^{d/2} \hat{f}(p)\hat{g}(p). \tag{2.45}$$

To prove the second formula, apply this to $\check{f} = \mathscr{F}^{-1} f$, $\check{g} = \mathscr{F}^{-1} g$ and use the inversion formula to find

$$\widehat{fg}(p) = (2\pi)^{-d/2} \check{f} * \check{g}(-p) = (2\pi)^{-d/2} \int \hat{f}(p+q)\hat{g}(-q)\mathrm{d}q = (2\pi)^{-d/2} \hat{f} * \hat{g}. \tag{2.46}$$

□

The important idea to keep in mind from this theorem is that $f * g$ inherits the regularity of the more regular of f, g and the decay of the more slowly decaying. In particular, $\mathscr{S}$ is invariant under convolution.

Corollary 2.15. *For $g \in \mathscr{S}(\mathbb{R}^d)$, the map $f \mapsto f * g$ maps $\mathscr{S}(\mathbb{R}^d)$ to itself and is continuous.*

Proof. The fact that $f * g \in \mathscr{S}(\mathbb{R}^d)$ follows from points *a), b)* of the previous theorem. Continuity follows from its proof, namely the bounds (2.40), (2.41) which show that $\|f * g\|_{\alpha,\beta}$ can be bounded in terms of $\|f\|_{\gamma,\beta}$ for finitely many γ. □

The convolution satisfies some important inequalities with respect to the norms on the Lebesgue spaces.

Lemma 2.16. *Let $g \in \mathscr{S}(\mathbb{R}^d)$ and $f \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, then*

$$\|f * g\|_p \le \|f\|_p \|g\|_1. \tag{2.47}$$

Proof. Let $q = p/(p-1)$ and write $|g(y)| = |g(y)|^{1/p}|g(y)|^{1/q}$. Then, by Hölder's inequality,

$$\int \left| \int f(x-y)g(y)\mathrm{d}y \right|^p \mathrm{d}x \le \int \int |f(x-y)|^p |g(y)| \mathrm{d}y\mathrm{d}x \Big(\int |g(z)|\mathrm{d}z \Big)^{p/q}$$
$$= \|f\|_p^p \|g\|_1^{1+p/q}. \tag{2.48}$$

Taking the p-th root and using again $1/p + 1/q = 1$ yields the claim.

□

These bounds show that, for example, the map $f \mapsto f * g$ for $g \in \mathscr{S}(\mathbb{R}^d)$ is continuous from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$, since it is bounded.

We will now use the convolution to approximate arbitrary elements of $L^p(\mathbb{R}^d)$ by smooth functions. We have already used point-wise approximation in the proof of the Fourier inversion formula, but now we are aiming at approximation in the L^p-norms. For this, we first need a Lemma concerning the action of translations on L^p.

Lemma 2.17. *Let $T_a f(x) = f(x-a)$ be the translation and $f \in L^p(\mathbb{R}^d)$ for $1 \le p < \infty$. Then*

$$\lim_{a\to 0} \|T_a f - f\|_p = 0.$$

Proof. We start by reducing the proof to the case $p = 1$. For this, note that

$$\lim_{R\to\infty} \|\chi_{B(0,R)} f - f\|_p = 0 \tag{2.49}$$

by Dominated Convergence, since $|\chi_{B(0,R)} f - f|^p \le |f|^p \in L^1(\mathbb{R}^d)$. Moreover, denoting by $A_R = \{x \in \mathbb{R}^d : ||f(x)| \le R\}$, we also have

$$\lim_{R\to\infty} \|\chi_{A_R} f - f\|_p = 0, \tag{2.50}$$

by the same reasoning. Hence, given $\varepsilon > 0$ we can find $R > 0$ and f_R with $\operatorname{supp} f_R \subset B(0,R)$ and $|f_R| \le R$ so that $\|f_R - f\|_p < \varepsilon$. Since T_a is an isometry, we thus have

$$\|T_a f - f\|_p \le 2\varepsilon + \|T_a f_R - f_R\|_p, \tag{2.51}$$

and

$$\int |f_R(x-a) - f_R(x)|^p \mathrm{d}x \le 2^{p-1} R^{p-1} \int |f_R(x-a) - f_R(x)|\mathrm{d}x, \tag{2.52}$$

which is finite because f_R has compact support. It is thus sufficient to prove the claim for $p = 1$.

Here, we need to use the fact from integration theory that any element of $L^1(\mathbb{R}^d)$ can be approximated in $L^1(\mathbb{R}^d)$ by a finite linear combination of characteristic functions on disjoint, half open rectangles [LL, Thm.1.18], that is, for $\varepsilon > 0$ there is

$$F = \sum_{i=1}^N \alpha_i \chi_{A_i} \tag{2.53}$$

with $\|F - f\|_1 < \varepsilon$. For any characteristic function χ_A of a rectangle

$$A = \{x \in \mathbb{R}^d : c_j < x_j \leq d_j\}, \tag{2.54}$$

$|\chi(x-a) - \chi(x)|$ is the characteristic function of the symmetric difference of A and its translate, whose volume is smaller than $2|a||\partial A|$, where $|\partial A|$ is the surface volume of A. We thus have

$$\|T_a f - f\|_1 \leq 2\varepsilon + 2|a| \sum_{j=1}^{N} |\alpha_i||\partial A_i|, \tag{2.55}$$

which is less than 3ε if $|a|$ is small enough. This proves the claim. □

Theorem 2.18 (Approximation by smooth functions)**.** *Let $g \in \mathscr{S}(\mathbb{R}^d)$ with $\int g = 1$ and set $g_n = n^d g(nx)$. For $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^d)$, the functions $f_n := f * g_n$ are smooth, and*

$$\lim_{n\to\infty} \|f_n - f\|_p = 0.$$

Proof. Smoothness of f_n is Theorem 2.14a). In order to prove the approximation in L^p, we write

$$f_n(x) - f(x) = \int (f(x-y) - f(x)) g_n(y) \mathrm{d}y. \tag{2.56}$$

Let $\varepsilon > 0$. We start by by considering the restriction of the integral to $|y| < \delta$. We have (cf. (2.48)),

$$\begin{aligned} \int \left| \int_{|y|<\delta} (f(x-y) - f(x)) g_n(y) \mathrm{d}y \right|^p \mathrm{d}x &\leq \int \int_{|y|<\delta} |f(x-y) - f(x)|^p |g_n(y)| \mathrm{d}y \mathrm{d}x \|g\|_1^{p-1} \\ &\leq \|g\|_1^p \sup_{|y|\leq\delta} \int |f(x-y) - f(x)|^p \mathrm{d}x. \end{aligned} \tag{2.57}$$

By Lemma 2.17, there exists δ so that $\|T_y f - f\|_p \leq \varepsilon$ for $|y| \leq \delta$, and we choose δ in this way. Then, we can find n_0 so that for $n \geq n_0$

$$\int_{|y|>\delta} |g_n(y)| = \int_{|y|>n\delta} |g(y)| \leq \varepsilon. \tag{2.58}$$

By the triangle inequality for the L^p norm (cf. Problem 10), we thus have by Lemma 2.16

$$\|f_n - f\|_p \leq \varepsilon + 2\|f\|_p \varepsilon \tag{2.59}$$

which proves the claim since ε was arbitrary. □

Corollary 2.19. *The Schwartz functions $\mathscr{S}(\mathbb{R}^d)$ are dense in $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$.*

Proof. Given $f \in L^p$ and $\varepsilon > 0$ we need to find $f_\varepsilon \in \mathscr{S}$ with $\|f - f_\varepsilon\|_p < \varepsilon$. To do this, first choose R so that $\|f - \chi_{B(0,R)} f\|_p < \varepsilon/2$. Then, let $f_{n,R} = \chi_{B(0,R)} f * g_n$ be a smooth approximation as in Theorem 2.18. This function is also rapidly decaying by Theorem 2.14b), since $\chi_{B(0,R)} f$ has compact support. Iterating this argument shows $f_{n,R} \in \mathscr{S}(\mathbb{R}^d)$. Choosing n large enough so that $\|f_{n,R} - \chi_{B(0,R)} f\| < \varepsilon/2$ yields the result. □

Remark 2.20. The Schwartz functions are clearly *not* dense in L^∞, since for the constant function $f \equiv 1$, $\|f - g\|_\infty = 1$ for all $g \in \mathscr{S}$, as g tends to zero for $|x| \to \infty$.

We can now define the Fourier transform on $L^2(\mathbb{R}^d)$, where it is a unitary map. Recall:

Definition 2.21. Let $A \in \mathrm{B}(\mathcal{H})$.

a) A is called *self-adjoint* if $A^* = A$;

b) A is called *unitary* if $A^* A = 1 = AA^*$;

c) A is called *normal* if $A^* A = AA^*$.

Theorem 2.22 (Fourier-Plancherel)**.** *The Fourier transform $\mathscr{F} : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ admits a unique continuous extension*

$$\overline{\mathscr{F}} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d).$$

This map is unitary, i.e.,

$$\langle f, \overline{\mathscr{F}} g\rangle = \langle \overline{\mathscr{F}^{-1}} f, g\rangle, \qquad \|f\|_2 = \|\overline{\mathscr{F}} f\|_2$$

for all $f, g \in L^2(\mathbb{R}^d)$ and $\overline{\mathscr{F}}^* = \overline{\mathscr{F}}^{-1} = \overline{\mathscr{F}^{-1}}$.

Proof. By Corollary 2.10, $\mathscr{F} : \mathscr{S}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defines a bounded linear map with norm one. Since $\mathscr{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, the unique continuous extension of this map $\overline{\mathscr{F}} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ also has norm one. The identities of Corollary 2.10 carry over to the extension by continuity of the scalar product and imply unitarity of the extension. □

Remark 2.23. The map $\overline{\mathscr{F}}$ is not given by the integral formula (2.1), which does not make sense for an arbitrary element of $L^2(\mathbb{R}^d)$. However, for $f \in L^1 \cap L^2(\mathbb{R}^d)$ this formula holds (by uniqueness of the extension). Then for any sequence $(f_n) \subset L^1 \cap L^2(\mathbb{R}^d)$ with $f_n \to f$ in $L^2(\mathbb{R}^d)$,

$$\overline{\mathscr{F}} f(x) = \lim_{n\to\infty} \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{-\mathrm{i}px} f_n(x) \mathrm{d}x. \tag{2.60}$$

For example, taking $f_n = \chi_{B(0,n)} f$,

$$\overline{\mathscr{F}} f(x) = \lim_{n\to\infty} \frac{1}{(2\pi)^{d/2}} \int_{|x|\le n} \mathrm{e}^{-\mathrm{i}px} f(x) \mathrm{d}x, \tag{2.61}$$

where convergence is in $L^2(\mathbb{R}^d)$.

Remark 2.24. The Fourier transform can be extended continuously to

$$\overline{\mathscr{F}} : L^p(\mathbb{R}^d) \to L^{\frac{p}{p-1}}(\mathbb{R}^d) \tag{2.62}$$

for $1 \leq p \leq 2$. However, this is in general not surjective, as can be seen from the case $p = 1$, where the range is contained in (but not equal to) $C_\infty(\mathbb{R}^d)$.

For the convolution map, the extension yields:

Corollary 2.25. *For $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^d)$ the convolution map*

$$\mathscr{S}(\mathbb{R}^d) \to L^p(\mathbb{R}^d), \qquad g \mapsto f * g$$

has a unique continuous extension to $L^1(\mathbb{R}^d)$ with norm $\|f\|_p$. The map

$$L^p(\mathbb{R}^d) \times L^1(\mathbb{R}^d) \to L^p(\mathbb{R}^d), \qquad (f, g) \mapsto f * g$$

is bilinear and continuous in each argument.

Proof. This follows directly from the bound (2.47) and the BLT Theorem. □

Question 2.26. Which of the following operators are normal and/or self-adjoint, unitary?

a) $M_g f = gf$ with $g \in L^\infty(\mathbb{R}^d)$ on $L^2(\mathbb{R}^d)$;

b) $T_v f = f(\cdot + v)$ with $v \in \mathbb{R}^d$ on $L^2(\mathbb{R}^d)$;

c) $T_t f = f(\cdot + t)$ with $t > 0$ on $L^2(\mathbb{R}_+)$.

Example 2.27. Let $f \in \mathscr{S}(\mathbb{R}^d)$ and $u(t, x)$ be the solution of the heat equation (see Example 2.12)

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x), & (t,x) \in (0,\infty) \times \mathbb{R}^d \\ \quad u(0,x) = f(x). \end{cases}$$

For $f \in \mathscr{S}(\mathbb{R}^d)$, $T_t f := u(t, \cdot)$, $t > 0$ can be written as (see Problem 16)

$$(T_t f)(x) = \int_{\mathbb{R}^d} E_t(x-y) f(y) \mathrm{d}y = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/(4t)} f(y) \mathrm{d}y. \tag{2.63}$$

The map $T_t : \mathscr{S}(R^d) \to \mathscr{S}(R^d)$ satisfies $\|T_t f\|_2 \leq \|f\|_2$ by Problem 16, so it extends continuously to a map $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ (still given by the formula (2.63)). This map is self-adjoint (and hence normal) on $L^2(\mathbb{R}^d)$, but not unitary for $t > 0$. Indeed, because $E(t, x-y)$ is real and symmetric under exchange of x, y, we have

$$\langle g, E_t * f \rangle = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \overline{g(x)} \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/(4t)} f(y) \mathrm{d}y \mathrm{d}x = \langle E_t * g, f \rangle, \tag{2.64}$$

so $T_t^* = T_t$. We have

$$T_t^* T_t = T_t^2 = T_{2t} \tag{2.65}$$

because $u(t+s)$ solves the heat equation with $u(t+0) = u(t)$. Now $T_{2t} \neq 1$ for $t > 0$ so T_t is not unitary. As $\|T_t f\| \leq \|f\|$, we say that T_t is a *contraction* (in fact, the inequality is strict because $\lambda = 1$ is not an eigenvalue of T_t, but the bound cannot be improved uniformly in f).

2.4. Duality and tempered distributions

We have now extended the Fourier transform to L^2, but we cannot use this for solving PDEs yet, since we most functions in L^2 are not differentiable in the standard sense. In order to overcome this, we will extend all the operations from $\mathscr{S}$ to a much larger space $\mathscr{S}'$ called the tempered distributions. This space also contains L^2, but not all of its elements can be thought of as functions.

Definition 2.28. Let X be a topological vector space. Then we define the topological dual of X by

$$X' := \{\varphi : X \to \mathbb{C},\ \varphi \text{ is linear and continuous}\}.$$

Example 2.29. If X is a Banach (i.e., normed, complete) space, then $X' = \mathrm{B}(X, \mathbb{C})$ is also a Banach space, with the norm

$$\|\varphi\|_{X'} = \sup_{0 \neq x \in X} \frac{|\varphi(x)|}{\|x\|}. \tag{2.66}$$

Example 2.30. If $X = \mathcal{H}$ is a Hilbert space, then $\mathcal{H}'$ and be indetified with $\mathcal{H}$ via the (anti-linear) isomorphism (cf. [FA, Thm.2.4.1])

$$\Phi : \mathcal{H} \to \mathcal{H}', \qquad \Phi(f)(g) \mapsto \langle f, g \rangle_{\mathcal{H}}. \tag{2.67}$$

Definition 2.31. Let X be a topological vector space. The weak topology on X' is the smallest topology so that for every $x \in X$ the evaluation

$$\iota_x : X' \to \mathbb{C}, \qquad \iota_x \varphi = \varphi(x) \tag{2.68}$$

is continuous. A sequence $(\varphi_n)_{n\in\mathbb{N}} \subset X'$ converges weakly to φ if

$$\forall x \in X : \lim_{n\to\infty} \varphi_n(x) = \varphi(x). \tag{2.69}$$

Definition 2.32. The space of tempered distributions is $\mathscr{S}'(\mathbb{R}^d) := (\mathscr{S}(\mathbb{R}^d))'$ equipped with the weak topology.

Remark 2.33. Since $\varphi \in \mathscr{S}'(\mathbb{R}^d)$ is linear, continuity is equivalent to continuity in $f = 0$, since if $f_n \to f$ in $\mathscr{S}$, then $f_n - f \to 0$, and

$$\lim_{n\to\infty} \varphi(f_n) = \varphi(f) \Leftrightarrow \lim_{n\to\infty} |\varphi(f_n) - \varphi(f)| = \lim_{n\to\infty} |\varphi(f_n - f)| = 0. \tag{2.70}$$

Question 2.34. Which of the following formulas define a tempered distribution on $\mathbb{R}$?

1. $f \mapsto f'(0)$,
2. $f \mapsto \int f^2(x)\mathrm{d}x$,
3. $f \mapsto \int e^{\sqrt{1+x^2}} f(x)\mathrm{d}x$,
4. $f \mapsto \int |x| f(x)\mathrm{d}x$.

Example 2.35. Let $g \in L^p(\mathbb{R}^d)$, $p \leq \infty$ then

$$f \mapsto \varphi_g(f) = \int_{\mathbb{R}^d} \overline{g}(x) f(x) \mathrm{d}x \tag{2.71}$$

defines an element of $\mathscr{S}'(\mathbb{R}^d)$. It is clearly linear, and if $f_n \to 0$ in $\mathscr{S}$, then by Hölder's inequality

$$|\varphi_g(f_n)| \leq C\|g\|_p \sum_{|\alpha|\leq 2d} \|f_n\|_{\alpha,0} \to 0, \tag{2.72}$$

so φ_g is continuous.

Many other classes of functions can be identified with tempered distributions by this formula.

Definition 2.36. A distribution $\varphi \in \mathscr{S}'(\mathbb{R}^d)$ is called a *regular distribution* if there exists $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ (i.e., $\chi_{B(0,R)} g \in L^1$ for all $R > 0$) such that $\varphi = \varphi_g$, that is

$$\forall f \in \mathscr{S}(\mathbb{R}^d) : \varphi(f) = \int_{\mathbb{R}^d} \overline{g}(x) f(x) \mathrm{d}x. \tag{2.73}$$

Proposition 2.37. *Let $\varphi = \varphi_g$ be a regular distribution, then g is unique. That is, if $h \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is such that $\varphi = \varphi_h$, then $h = g$ almost everywhere.*

Proof. We have to show that

$$\forall \in \mathscr{S}(\mathbb{R}^d) : \int \overline{g}(x) f(x) \mathrm{d}x = \int \overline{h}(x) f(x) \mathrm{d}x \implies g = h \text{ a.e..} \tag{2.74}$$

By additivity in in g, h we may consider $\eta = g - h$ and show that $\varphi_\eta = 0$ implies $\eta = 0$ a.e.. By choosing f of compact support, we may assume that $\eta \in L^1$, without loss of generality. Now let $g_n(x) = n^d \mathrm{e}^{-n^2 x^2/2} \in \mathscr{S}(\mathbb{R}^d)$. By hypothesis, for every $x \in \mathbb{R}^d$

$$g_n * \overline{\eta}(x) = \int \overline{\eta}(y) g_n(x-y) \mathrm{d}y = \varphi_\eta(g_n(x - \cdot)) = 0. \tag{2.75}$$

On the other hand, by Theorem 2.18, $g_n * \overline{\eta}$ converges to $\overline{\eta}$ in L^1, so $\eta = 0$ in L^1 and thus almost everywhere. □

We can extend many (linear) operations on $\mathscr{S}$ to $\mathscr{S}'$ by duality, i.e. taking the transpose.

Proposition 2.38. *Let X, Y be topological vector spaces and $T : X \to Y$ a linear, continuous map. Then there exists a unique weakly continuous map $T' : Y' \to X'$ satisfying*

$$(T'\varphi)(x) = \varphi(Tx).$$

Proof. The formula defines a unique map since $\varphi \in X'$ is completely determined by its evaluations. This map is linear, since

$$T'(a\varphi + \psi)(x) = (a\varphi + \psi)(Tx) = a\varphi(Tx) + \psi(Tx) = aT'\varphi(x) + T'\psi(x). \tag{2.76}$$

It is weakly (sequentially) continuous since

$$\lim_{n\to\infty} T'\varphi_n(x) = \lim_{n\to\infty} \varphi_n(Tx) = \varphi(Tx) = T'\varphi(x) \tag{2.77}$$

for any weakly convergent sequence $\varphi_n \overset{w}{\to} \varphi$. □

Examples 2.39.

a) Fourier transform $\mathscr{F}$. For $g \in \mathscr{S}(\mathbb{R}^d)$ we have

$$((\mathscr{F}^{-1})'\varphi_g)(f) = \varphi_g(\mathscr{F}^{-1}f) = \int \overline{g}(x)(\mathscr{F}^{-1}f)(x)\mathrm{d}x \overset{\text{Parseval}}{=} \int \overline{\hat{g}(p)} f(p)\mathrm{d}p = \varphi_{\hat{g}}(f), \tag{2.78}$$

so the action of $(\mathscr{F}^{-1})'$ on $\mathscr{S}'$ extends the one of $\mathscr{F}$ on $\mathscr{S}$. We will also denote this by

$$(\mathscr{F}^{-1})'\varphi = \mathscr{F}\varphi =: \hat{\varphi}. \tag{2.79}$$

b) Derivative: For any $\alpha \in \mathbb{N}^d$ we have $(\partial^\alpha)' : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ linear and continuous. In this way we can define derivatives of all tempered distributions, in particular all L^2-functions.

c) Multiplication by a monomial: In this case we have $(x^\alpha)'\varphi_g = \varphi_{x^\alpha g} =: x^\alpha \varphi_g$.

d) Convolution with a Schwartz function. For fixed $g \in \mathscr{S}(\mathbb{R}^d)$, the map

$$f \mapsto g * f \tag{2.80}$$

is linear and continuous on $\mathscr{S}(\mathbb{R}^d)$. It thus extends to $\mathscr{S}'(\mathbb{R}^d)$. For suitable h, the formula

$$(g*)'\varphi_h(f) = \varphi_h(g * f) = \int \overline{h}(x) \int g(x-y) f(y)\mathrm{d}y\mathrm{d}x = \varphi_{h*Cg}(f) \tag{2.81}$$

holds with $Cg(x) := \overline{g}(-x)$. We thus define the convolution of g with $\varphi \in \mathscr{S}'(\mathbb{R}^d)$ as

$$g *_{\mathscr{S}'} \varphi := (Cg*)'\varphi(f). \tag{2.82}$$

Definition 2.40. Let $\alpha \in \mathbb{N}^d$. The α-th *distributional derivative* on $\mathscr{S}'(\mathbb{R}^d)$ is defined as $(\partial^\alpha)_{\mathscr{S}'} := (-1)^{|\alpha|}(\partial^\alpha)'$.

Remark 2.41. The definition of $(\partial^\alpha)_{\mathscr{S}'}$ ensures that its action is compatible with the usual derivative and integration by parts: For $g \in \mathscr{S}(\mathbb{R}^d)$

$$((\partial^\alpha)_{\mathscr{S}'}\varphi_g)(f) = \int \overline{g(x)}(-1)^{|\alpha|}\partial_x^\alpha f(x)\mathrm{d}x = \int (\partial_x^\alpha \overline{g})(x) f(x)\mathrm{d}x = \varphi_{\partial^\alpha g}(f). \qquad (2.83)$$

For this reason we will not distinguish $(\partial^\alpha)_{\mathscr{S}'}$ from the usual derivative by the notation. The distributional derivative is a local operation: Let $\varphi \in \mathscr{S}'$ have support in the open set $\Omega \subset \mathbb{R}^d$ (i.e.: $\operatorname{supp} f \subset \Omega^c \implies \varphi(f) = 0$), then $\operatorname{supp} \partial^\alpha \varphi \subset \Omega$.

Also note that

$$(\mathscr{F}\partial^\alpha\varphi)(f) = \varphi\left((-1)^{|\alpha|}\partial^\alpha \mathscr{F}^{-1} f\right) = \varphi\left(\mathscr{F}^{-1}(-\mathrm{i})^{|\alpha|}p^\alpha f\right) = \left((-\mathrm{i})^{|\alpha}|p^\alpha \mathscr{F}\varphi\right)(f), \qquad (2.84)$$

where multiplication by p^α is defined as M'_{p^α} and in the last step, we used linearity of $f \mapsto \varphi(f)$. Since $g \mapsto \varphi_g$ is anti-linear, this means

$$\varphi_{(\mathrm{i}p)^\alpha \hat{g}} = (-\mathrm{i}p)^\alpha \varphi_{\hat{g}} = \widehat{\partial^\alpha \varphi_g} = \varphi_{\widehat{\partial^\alpha g}}, \qquad (2.85)$$

which is consistent with the formula from Proposition 2.9.

2.5. Elliptic PDEs and Sobolev spaces

We can now solve equations such as

$$(\Delta + z)u = f$$

even with $f \in \mathscr{S}'$ by the Fourier transform method (cf. Example 2.11). However, at first we only know that the solution u is an element of $\mathscr{S}'$. We do not, for instance, have a criterion that tells us if $u \in C^k$ and we have found a classical solution.

It is thus important to investigate further these (distributional) solutions. For a special class of constant coefficient linear PDEs, called *elliptic* this can be done quite easily and the regularity of solutions is described precisely by the *Sobolev spaces*.

Definition 2.42. Let

$$P = \sum_{|\alpha| \le k} a_\alpha \partial^\alpha \qquad (2.86)$$

be a constant-coefficient differential operator of order k. The *symbol* of P is the function

$$\sigma_P(p) := \sum_{|\alpha| \le k} a_\alpha (ip)^\alpha.$$

Since

$$\mathscr{F}Pu = \sigma_P \mathscr{F}u, \qquad (2.87)$$

we can solve PDEs as in Example 2.11 if σ_P is invertible for every k. However, the regularity can still be difficult to analyse. The following condition simplifies this enormously:

Definition 2.43. A constant-coefficient differential operator of order k is called *uniformly elliptic* if there exists $c > 0$ so that for all $p \in \mathbb{R}^d$

$$\sum_{|\alpha|=k} a_\alpha (ip)^\alpha \geq c|p|^k. \tag{2.88}$$

We note that this can only hold if $k = 2m$ is even, and only concerns the terms of the highest order in P. The terminology comes from the second order case, where the the condition means that the level sets of σ_P are ellipses.

We will now focus on the simplest elliptic operator $P = -\Delta$, $\sigma_P(p) = -(ip)^2 = p^2$. With some care, results for the general case can be obtained by the same arguments. Our goal is to show that if u is a solution to

$$-\Delta u = f \tag{2.89}$$

and $f \in C^m$, then $u \in C^n$ for an appropriate n (which will depend on the dimension).

Since our method relies on the Fourier transform and this is naturally defined in $\mathscr{S}$, $\mathscr{S}'$ and not C^m, we first need to study subspaces of $\mathscr{S}'$ that classify the regularity of distributions.

Definition 2.44. Let $s \in \mathbb{R}$. The *Sobolev space* of order s is the space

$$H^s(\mathbb{R}^d) := \left\{ \varphi \in \mathscr{S}'(\mathbb{R}^d) : \hat{\varphi} \text{ is regular, and } (1 + |\cdot|^2)^{s/2}\hat{\varphi} \in L^2(\mathbb{R}^d) \right\} \tag{2.90}$$

with the norm

$$\|\varphi\|_{H^s} = \left\| (1 + |\cdot|^2)^{s/2}\hat{\varphi} \right\|_{L^2}. \tag{2.91}$$

Remark 2.45. Note that for any $s \in \mathbb{R}$ the Sobolev space $H^s(\mathbb{R}^d)$ is a Hilbert space with the scalar product

$$\langle f, g \rangle_{H^s} = \int (1 + p^2)^s \overline{\hat{f}(p)} \hat{g}(p) \mathrm{d}p.$$

That ist, the norm satisifes $\|f\| := \sqrt{\langle f, f \rangle}$, and H^s is complete with this norm, because

$$\varphi \mapsto (1 + |\cdot|^2)^{s/2}\hat{\varphi} \tag{2.92}$$

is an isometry from H^s to L^2, which is complete.

Definition 2.46. An orthonormal system (ONS) in $\mathcal{H}$ is a family $\{e_i, i \in I\} \subset \mathcal{H}$, such that

$$\langle e_i, e_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

An orthonormal system is called complete (or an orthonormal Hilbert basis) if for every $f \in \mathcal{H}$

$$f = \sum_{i \in I} \langle e_i, f \rangle e_i. \tag{2.93}$$

A Hilbert space is called separable if there exists a countable complete ONS in $\mathcal{H}$. In the following we will only consider separable Hilbert spaces.

Example 2.47. The Sobolev spaces $H^s(\mathbb{R}^d)$ are separable. A complete ONS can be given as follows. First, choose a complete ONS in $L^2(\mathbb{R}^d)$, for example the Hermite functions (here in $d = 1$)

$$e_n(x) := c_n H_n(x) e^{-\frac{1}{2}x^2}, \tag{2.94}$$

where $H_n(x)$ are the Hermite polynomials, c_n normalizing constants, and $n \in \mathbb{N}_0$. A complete ONS of $H^s(\mathbb{R}^d)$ is then given by $e_{n,s}(x) := \mathscr{F}^{-1}(1+p^2)^{-s/2}\hat{e}_n(p)$.

Remark 2.48. Note that separability does not mean that the vector-space dimension of $\mathcal{H}$ is countable (that would require the linear combination to be finite). In fact, the vector space dimension of a Hilbert space is either finite or uncountable by Baire's theorem.

Proposition 2.49.

a) We have $H^s(\mathbb{R}^d) \subset H^t(\mathbb{R}^d)$ for $s \geq t$, and in particular $H^s(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ for all $s \geq 0$.

b) If $s \in \mathbb{N}$ is a non-negative integer, then $f \in H^s(\mathbb{R}^d)$ if and only if $f \in L^2(\mathbb{R}^d)$ and $\partial^\alpha f \in L^2(\mathbb{R}^d)$ for all $|\alpha| \leq s$.

Proof. a): Let $s \geq t$. Then

$$\frac{(1+p^2)^t}{(1+p^2)^s} \leq C \tag{2.95}$$

for some $C > 0$. Thus for $f \in H^s$ we have $(1+p^2)^{t/2}\hat{f} \in L^2$, because

$$\begin{aligned}\int (1+p^2)^t |\hat{f}(p)|^2 \mathrm{d}p &= \int \frac{(1+p^2)^t}{(1+p^2)^s}(1+p^2)^s|\hat{f}(p)|^2\mathrm{d}p \\ &\leq C\int (1+p^2)^s|\hat{f}(p)|^2\mathrm{d}p = C\|f\|^2_{H^s}.\end{aligned} \tag{2.96}$$

Hence $f \in H^t$ and thus $H^s \subset H^t$. As $H^0 = L^2$ by definition this proves a).

b): Let first $f \in H^m(\mathbb{R}^d)$, $m \in \mathbb{N}$. Then $f \in L^2$ by a) and we have for the derivative in $\mathscr{S}'$

$$\partial^\alpha f = \mathscr{F}^{-1}(\mathrm{i}p)^\alpha \hat{f}. \tag{2.97}$$

By Plancherel's Theorem it is thus enough to show that $(ip)^\alpha \hat{f} \in L^2$ for $|\alpha| \leq m$. This now follows from the inequalities

$$\left|(\mathrm{i})^{|\alpha|}p_1^{\alpha_1}\cdots p_d^{\alpha_d}\hat{f}(p)\right|^2 \leq |p|^{2|\alpha|}|f(p)|^2 \leq (1+p^2)^{|\alpha|}|f(p)|^2 \leq (1+p^2)^m|f(p)|^2. \tag{2.98}$$

For the reverse implication, we have by Plancherel that $(ip)^\alpha \hat{f}$ for all $|\alpha| \leq m$ and thus $p^{2\alpha}\hat{|}f(p)|^2 \in L^1$. Now

$$p^{2m} = (p_1^2 + \cdots + p_d^2)^m = \sum_{|\alpha|=m}\frac{m!}{\alpha!}p^{2\alpha} \tag{2.99}$$

by the multinomial theorem, so $p^{2m}|\hat{f}(p)|^2 \in L^1$. This implies that $(1+p^2)^{m/2}\hat{f} \in L^2$ because $(1+p^{2m})/(1+p^2)^m$ is bounded, by the argument of (2.96)

□

By the Riesz Representation theorem, the dual $(H^s)'$ can be identified with H^s itself. However, it is more natural to identify it with H^s.

Proposition 2.50. *Let $\varphi \in \mathscr{S}'(\mathbb{R}^d)$, $s \in \mathbb{R}$ and assume that there exists a constant $C \geq 0$ so that for all $f \in \mathscr{S}(\mathbb{R}^d)$*

$$|\varphi(f)| \leq C\|f\|_{H^s}.$$

Then there exists $h \in H^{-s}(\mathbb{R}^d)$ so that

$$\varphi(f) = \int \overline{\hat{h}(p)}\hat{f}(p)\mathrm{d}p.$$

Proof. By the assumed inequality, the linear map $\varphi : \mathscr{S} \to \mathbb{C}$ is bounded with respect to the H^s-norm. Since $\mathscr{S}$ is dense in H^s (Problem **??**), we can thus extend φ uniquely to a continuous linear functional on $H^s(\mathbb{R}^d)$ by the B.L.T Theorem. By the Riesz Representation Theorem, there exists $g \in H^s(\mathbb{R}^d)$ so that

$$\varphi(f) = \int (1+p^2)^s\overline{g(p)}\hat{f}(p)\mathrm{d}p = \int \overline{\hat{h}(p)}\hat{f}(p)\mathrm{d}p \tag{2.100}$$

with $\hat{h}(p) := (1+p^2)^s\hat{g}(p)$, which is clearly an element of $H^{-s}(\mathbb{R}^d)$. □

A. Appendix

A.1. The Lebesgue integral

This section summarizes those results from the theory of integration that are most important for the course, see [Ru] for an introduction and [LL] for more details.

Let $\mathscr{B}(\mathbb{R}^d)$ be the Borel σ-algebra on $\mathbb{R}^d$. That is, the smallest collection of subsets $B \subset \mathbb{R}^d$ that contains all open sets and is closed under complements, finite intersections and countable unions. Elements of $\mathscr{B}$ are called measurable sets.

Definition A.1. A measure is a function

$$\mu : \mathscr{B}(\mathbb{R}^d) \to \mathbb{R}_+ \cup \{\infty\}$$

with the properties

$$\mu(\emptyset) = 0$$
$$\mu\Big(\bigcup_{j=1}^{\infty} B_j\Big) = \sum_{j=1}^{\infty} \mu(B_j)$$

for any family of disjoint sets $(B_j)_{j\in\mathbb{N}}$.

The Lebesgue measure λ is the unique measure that is invariant by translation and satisfies $\lambda([0,1]^d) = 1$.

Definition A.2. A function $f : \mathbb{R}^d \to \mathbb{C}$ is called measureable if for every $B \in \mathscr{B}(\mathbb{C}) \cong \mathscr{B}(\mathbb{R}^2)$

$$f^{-1}(B) = \{x \in \mathbb{R}^d : f(x) \in B\}$$

is measurable, i.e., an element of $\mathscr{B}(\mathbb{R}^d)$.

The characteristic function χ_B of any set $B \in \mathscr{B}(\mathbb{R}^d)$ is measurable. Its integral is defined as

$$\int \chi_B(x)\lambda(\mathrm{d}x) = \lambda(B). \tag{A.1}$$

A simple function is a linear combination of characteristic functions. Any measurable function is the pointwise limit of simple functions,

$$f(x) = \lim_{n\to\infty} \sum_{j=1}^{n} a_{j,n}\chi_{B_{j,n}}(x). \tag{A.2}$$

Moreover, if f is *non-negative*, the simple functions can be chosen so that the value in each point is increasing in n. For a non-negative function one thus defines

$$\int f(x)\lambda(\mathrm{d}x) := \lim_{n\to\infty} \sum_{j=1}^{n} a_{j,n}\lambda(B_{j,n}) \in \mathbb{R}_+ \cup \{\infty\}. \tag{A.3}$$

Since the right hand side is an increasing sequence of numbers that are positive or $+\infty$, this is well defined but possibly infinite.

Definition A.3. A positive measurable function $f : \mathbb{R}^d \to \mathbb{R}_+$ is called integrable if (A.3) is finite.

A measurable function $f : \mathbb{R}^d \to \mathbb{C}$ is called integrable if $|f|$ is integrable.

If $f : \mathbb{R}^d \to \mathbb{C}$ is integrable, then

$$\int f(x)\mathrm{d}x = \int f(x)\lambda(\mathrm{d}x) = \lim_{n\to\infty} \sum_{j=1}^{n} a_{j,n}\lambda(B_{j,n}) \tag{A.4}$$

is a well-defined complex number.

If $A \in \mathscr{B}(\mathbb{R}^d)$ is a measurable set we define

$$\int_A f(x)\mathrm{d}x = \int \chi_A(x)f(x)\mathrm{d}x, \tag{A.5}$$

where χ_A is the characteristic function. We say that f is integrable on A if $f\chi_A$ is integrable.

If f is Riemann-integrable then f is Lebesgue-integrable and the integrals are equal [Ru, Thm.11.33].

Definition A.4 (Lebesgue spaces)**.** Let $1 \leq p < \infty$

$$\mathscr{L}^p(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C} : |f|^p \text{ is integrable}\}.$$

The Lebesgue space $L^p(\mathbb{R}^d)$ is the quotient of $\mathscr{L}^p(\mathbb{R}^d)$ under the equivalence relation

$$f \sim g :\Leftrightarrow \lambda\Big(\{x : f(x) \neq g(x)\}\Big) = 0$$

of equality almost everywhere. It is a Banach space with the norm

$$\|f\|_p = \Big(\int |f|^p(x)\mathrm{d}x\Big)^{1/p},$$

where f is any representative in the equivalence class.

For $p = \infty$ we define $\mathscr{L}^p(\mathbb{R}^d)$ as the space of measureable functions for which

$$\|f\|_\infty = \text{ess}-\text{sup}|f| := \inf\Big\{t \in \mathbb{R} : \lambda(f^{-1}(t,\infty)) = 0\Big\} \tag{A.6}$$

is finite. The Lebesgue space $L^p(\mathbb{R}^d)$ is the quotient of $\mathscr{L}^p(\mathbb{R}^d)$ by the same equivalence relation.

Proposition A.5 (Hölder's inequality)**.** *Let $1 \leq p, q \leq \infty$ so that $p^{-1} + q^{-1} = 1$, with the convention that $\infty^{-1} = 0$. Then for $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ we have $fg \in L^1(\mathbb{R}^d)$ and*

$$\left| \int f(x)g(x)\mathrm{d}x \right| \leq \|f\|_p \|g\|_q. \tag{A.7}$$

For $d > 1$ an important result concerns the relation of the d-dimensional integral and the iteration of lower-dimensional integrals.

Theorem A.6. *Fubini-Tonelli Let $n, m \geq 1$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ be a measurable function and $A \in \mathscr{B}(\mathbb{R}^{n+m})$.*

a) If $f \geq 0$, then

$$\int_A f(x,y)\lambda(\mathrm{d}(x,y)) = \int_{\pi_1(A)} \Bigg(\int_{\pi_1^{-1}(\{x\})\cap A} f(x,y)\mathrm{d}y \Bigg) \mathrm{d}x = \int_{\pi_2(A)} \Bigg(\int_{\pi_2^{-1}(\{y\})\cap A} f(x,y)\mathrm{d}x \Bigg) \mathrm{d}y$$

where $\pi_j(A)$, $j = 1, 2$ are the projections of A to $\mathbb{R}^n$, $\mathbb{R}^m$ respectively, and the equality is understood in the sense that if one expression is infinite, all are.

b) If f is integrable on A, then

a) The functions

$$x \mapsto f(x,y), \qquad y \mapsto f(x,y)$$

are integrable on $\pi_2^{-1}(\{y\})\cap A$ for almost every $y \in \mathbb{R}^m$, respectively on $\pi_1^{-1}(\{x\})\cap A$ for almost every $x \in \mathbb{R}^n$;

b) the functions (set equal to zero where the integral is not defined)

$$\varphi(y) = \int_{\pi_2^{-1}(\{y\})\cap A} f(x,y)\mathrm{d}x, \qquad \psi(x) = \int_{\pi_1^{-1}(\{x\})\cap A} f(x,y)\mathrm{d}y$$

are integrable;

c) the identity

$$\int_{\pi_2(A)} \varphi(y)\mathrm{d}y = \int_A f(x,y)\mathrm{d}y = \int_{\pi_1(A)} \psi(x)\mathrm{d}x$$

holds.

The well-known transformation formula holds for the Lebesgue integral.

Theorem A.7 (Change of variables)**.** *Let $A \in \mathscr{B}(\mathbb{R}^d)$, let $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be a C^1-diffeomorphism, and denote by $|J(x)| := |\det D\varphi(x)|$. Then if f is integrable on A, $x \mapsto f(\varphi(x))|J(x)|$ is integrable on $\varphi^{-1}(A)$ and*

$$\int_A f(x)\mathrm{d}x = \int_{\varphi^{-1}(A)} f(\varphi(x))|J(x)|\mathrm{d}x.$$

The most important properties of the Lebesgue integral are the convergence theorems.

Theorem A.8 (Monotone Convergence)**.** *Let $(f_n)_n \in \mathbb{N}$ be a sequence of measurable functions with $f_n \leq f_{n+1}$ and*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

almost everywhere for some function $f : \mathbb{R}^d \to \mathbb{C}$. Then f is measureable and

$$\lim_{n\to\infty} \int f_n(x)\mathrm{d}x = \int f(x)\mathrm{d}x.$$

Theorem A.9 (Dominated Convergence)**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions and assume there is a measurable function f so that*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

almost everywhere. Assume moreover that there exists a positive, integrable function g so that

$$\forall n \in \mathbb{N} : |f_n| \leq g$$

almost everywhere. Then f is integrable and

$$\lim_{n\to\infty} \int f_n(x)\mathrm{d}x = \int f(x)\mathrm{d}x.$$

An important corollary to this result concerns the exchange of integration and differentiation.

Corollary A.10. *Let $U \subset \mathbb{R}^k$ be open and $f : U \times \mathbb{R}^d \to \mathbb{C}$ a measurable function such that*

1. *for all $\eta \in U$, $x \mapsto f(\eta, x)$ is integrable,*
2. *for almost all $x \in \mathbb{R}^d$, $\eta \mapsto f(\eta, x)$ is continuously differentiable,*
3. *there exists a positive, integrable function $g : \mathbb{R}^d \to \mathbb{R}_+$ with*

$$\forall \eta \in U : \left|\nabla_\eta f(\eta, x)\right| \leq g(x).$$

Then $\eta \mapsto \int f(\eta, x)\mathrm{d}x$ is continuously differentiable and for all $j = 1, \dots, k$

$$\partial_{\eta_j} \int f(\eta, x)\mathrm{d}x = \int \partial_{\eta_j} f(\eta, x)\mathrm{d}x.$$

B. Problems

Problem 1. Part 1
For any $n \in \mathbb{N}$, we set $f_n := \mathbf{1}_{[n,n+1]}$.

1. Show that for any $x \in \mathbb{R}_+$, $\lim_{n\to+\infty} f_n(x) = 0$

2. Show that for any $n \in \mathbb{N}$, we have $\int_{\mathbb{R}_+} f_n(x)dx = 1$

Part 2
We will show that the sequence $(f_n)_{n\in\mathbb{N}}$ does not satisfy the following property: there exist a non-negative function $g \in L^1(\mathbb{R}_+)$ such that

$$\text{a.e. } x \in \mathbb{R}_+, \ \forall n \in \mathbb{N}, \quad |f_n(x)| \leq g(x). \tag{B.1}$$

1. Show that for any $x \in \mathbb{R}_+$
$$\sup_{n\in\mathbb{N}}\{|f_n(x)|\} = 1.$$

2. Show that, if a measurable function $g : \mathbb{R}_+ \to \mathbb{R}$ satisfying (B.1), then $g \notin L^1(\mathbb{R}_+)$.

Problem 2. Let $a \in \mathbb{C}$ such that $\mathrm{Re}(a) > 0$. The goal of this exercise is to show that

$$\forall x \in \mathbb{R}^d, \quad \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-\frac{|x|^2}{2a}} dx = a^{\frac{d}{2}} e^{-\frac{a}{2}|\xi|^2} \tag{B.2}$$

Part 1
For any $x \in \mathbb{R}$, we define $h(x) := e^{-\frac{x^2}{2a}}$. We assume that $h \in \mathscr{S}(\mathbb{R}^d)$.

1. Show that $h'(x) = -\frac{x}{a}h(x)$.

2. Show that $h^{'} \in L^1(\mathbb{R})$ and that $\widehat{h'}(\xi) = i\xi\widehat{h}(\xi)$.

3. Show that $\widehat{h}'(\xi) = -i\widehat{xh}(\xi)$.

4. Recall that
$$\int_{\mathbb{R}} h(x)dx = \sqrt{2a\pi}.$$
Show that $\widehat{h}(0) = \sqrt{a}$.

5. Deduce that $\widehat{h}$ is the solution of the following Cauchy problem
$$\begin{cases} \widehat{h}'(\xi) = -a\xi\widehat{h}(\xi) \ \text{ in } \mathbb{R}, \\ \widehat{h}(0) = \sqrt{a}. \end{cases} \tag{B.3}$$

6. Deduce from that, for any $\xi \in \mathbb{R}$

$$\widehat{h}(\xi) = \sqrt{a}e^{-\frac{a}{2}\xi^2}.$$

Part 2 By remarking that for any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have

$$e^{-\frac{|x|^2}{2a}} = \prod_{j=1}^{d} h(x_j),$$

show Formula (B.2).

Problem 3. Let $u_0 \in \mathscr{S}(\mathbb{R}^d)$. For any $t \geq 0$ and $\xi \in \mathbb{R}^d$, we set

$$u(t,x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t|\xi|^2} \widehat{u}_0(\xi) d\xi.$$

Part 1

1. Show that for any $(t,x) \in (0,+\infty) \times \mathbb{R}^d$, we have

$$\partial_t u(t,x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} (-|\xi|^2) e^{ix\cdot\xi} e^{-t|\xi|^2} \widehat{u}_0(\xi) d\xi.$$

2. Show that $u \in C^\infty((0,+\infty) \times \mathbb{R}^d)$.
3. Show that $\partial_t u - \Delta u = 0$ in $(0,+\infty) \times \mathbb{R}^d$.

Part 2

1. Show that $(t,x) \in (0,+\infty) \times \mathbb{R}^d$,

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t|\xi|^2} \widehat{u}_0(\xi) d\xi = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy.$$

2. Show that $\lim_{t\to 0^+} u(t,x) = u_0(x)$.
3. Deduce that for any $x \in \mathbb{R}^d$, we have $u(0,x) = u_0(x)$.

Part 3
Show that, for any $f \in \mathscr{S}(\mathbb{R}^d)$, we have

$$f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi) d\xi.$$

Problem 4. For multiindeces $\alpha, \beta \in \mathbb{N}^d$, we declare that $\beta \leq \alpha$ if $\beta_j \leq \alpha_j$ for all $j = 1, \ldots, d$. Denote by

$$\binom{\alpha}{\beta} = \prod_{j=1}^{d} \binom{\alpha_j}{\beta_j}.$$

Prove the generalised Leibniz formula for $f, g \in C^{|\alpha|}(\mathbb{R}^d)$

$$\partial^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g).$$

Problem 5. Let $u_0 \in \mathscr{S}(\mathbb{R}^d)$. For any $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ we set

$$u(t,x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{it|\xi|^2} \widehat{u}_0(\xi) d\xi.$$

a) Show that $u \in C^\infty(\mathbb{R} \times \mathbb{R}^d)$.

b) Show that u solves the Schrödinger equation

$$\begin{cases} \partial_t u + i\Delta u = 0, & \text{in } \mathbb{R} \times \mathbb{R}^d, \\ \lim_{t\to 0} u(t,x) = u_0(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{B.4}$$

Problem 6. Let u_0 and u_1 in $\mathscr{S}(\mathbb{R}^d)$. For any $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ we set

$$u(t,x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \cos(t|\xi|) \widehat{u}_0(\xi) d\xi + \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \frac{\sin(t|\xi|)}{|\xi|} \widehat{u}_1(\xi) d\xi.$$

1. Show that $u \in C^\infty(\mathbb{R} \times \mathbb{R}^d)$.
2. Show that u solves the wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & \text{in } \mathbb{R} \times \mathbb{R}^d, \\ \lim_{t\to 0} u(t,x) = u_0(x) \quad \text{and} \quad \lim_{t\to 0} \partial_t u(t,x) = u_1(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{B.5}$$

Problem 7. Let f and g in $\mathscr{S}(\mathbb{R}^d)$ and P be a polynomial function. Show the following properties

- $fg \in \mathscr{S}(\mathbb{R}^d)$,
- $Pf \in \mathscr{S}(\mathbb{R}^d)$.

Problem 8. Let $u_0 \in \mathscr{S}(\mathbb{R}^d)$ and $v \in \mathbb{R}^d$.

1. Let us set for any $(t,\xi) \in \mathbb{R} \times \mathbb{R}^d$, $\Phi(t,\xi) := e^{itv\cdot\xi} \widehat{u}_0(\xi)$.
   a) Show that for any $t \in \mathbb{R}$, $\Phi(t,\cdot) \in \mathscr{S}(\mathbb{R}^d)$.
   b) Show that the function $u : (t,x) \in \mathbb{R} \times \mathbb{R}^d \mapsto u(t,x) := \mathcal{F}^{-1}(\Phi(t,\cdot))(x)$ satisfies
   $$\begin{cases} \partial_t u - v \cdot \nabla u = 0 & \text{in } \mathbb{R} \times \mathbb{R}^d, \\ u(0,\cdot) = u_0 & \text{in } \mathbb{R}^d. \end{cases}$$
2. Using the Fourier inversion formula, find $\varphi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ such that for any $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, we have $u(t,x) = u_0(\varphi(t,x))$.
3. Let $p \in [1,+\infty]$. Show that
$$\forall t \in \mathbb{R}, \quad \|u(t,\cdot)\|_{L^p} = \|u_0\|_{L^p}.$$

Problem 9. Let p, q and r in $[1,\infty]$ such that

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}.$$

Let $f\in L^p(\mathbb{R}^d)$ and $g\in L^q(\mathbb{R}^d)$. The goal of this exercise is to show that

$$\|fg\|_{L^r}\leq\|f\|_{L^p}\|g\|_{L^q}. \tag{B.6}$$

1. Show (B.6) for $r=\infty$.
2. Assume that $r\neq\infty$. Deduce (B.6) from the standard Hölder estimate (which correspond to the case $r=1$).
   Hint: use that $r/p+r/q=1$.

Problem 10. Let $p\in[1,\infty]$ and $g,f\in L^p(\mathbb{R}^d)$. The goal is to show that

$$\|f+g\|_{L^p}\leq\|f\|_{L^p}+\|g\|_{L^p}. \tag{B.7}$$

1. Show (B.7) for $p=\infty$ and $p=1$.
2. Assume that $p\in]1,\infty[$.
   a) Show that
   $$|f(x)+g(x)|^p\leq|f(x)||f(x)+g(x)|^{p-1}+|g(x)||f(x)+g(x)|^{p-1}.$$
   b) Show that
   $$\int_{\mathbb{R}^d}|f(x)||f(x)+g(x)|^{p-1}dx\leq\|f\|_{L^p}\|f+g\|_{L^p}^{\frac{p-1}{p}}.$$
   c) Deduce (B.7).

Problem 11. Let $p\in[1,\infty[$. Show that

$$\forall\lambda>0,\quad\int_{\mathbb{R}^d}\mathbf{1}_{\{|f|\geq\lambda\}}dx\leq\frac{1}{\lambda^p}\|f\|_{L^p}^p.$$

Problem 12. Let p, q and r in $[1,\infty]$ such that

$$\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}.$$

Let $f\in L^p(\mathbb{R}^d)$ and $g\in L^q(\mathbb{R}^d)$. The goal of this exercise is to show that $f\star g\in L^r(\mathbb{R}^d)$, with

$$\|f\star g\|_{L^r}\leq\|f\|_{L^p}\|g\|_{L^q}. \tag{B.8}$$

1. Assume that $r=\infty$. Show that (B.8) holds.
2. Assume that $p=q=1$. Show that (B.8) holds.

3. Assume that $p = 1$.
   a) Show that
   $$\left(\int_{\mathbb{R}^d} |f(x-y)||g(y)|dy\right)^q \leq \left(|f| \star |g|^q\right)(x)\|f\|_{L^1}^{q-1}.$$
   Hint: Remark that $|f(x-y)||g(y)| = |f(x-y)|^{1-\frac{1}{q}}|f(x-y)|^{\frac{1}{q}}|g(y)|$.
   b) Deduce from 2. that
   $$\|f \star g\|_{L^q} \leq \|f\|_{L^1}\|g\|_{L^q}.$$
4. Assume that p, q and r belong to $]1,\infty[$.
   a) Let p_1, p_2 and p_3 in $[1,\infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ and $u \in L^{p_1}(\mathbb{R}^d)$, $v \in L^{p_2}(\mathbb{R}^d)$ and $w \in L^{p_3}(\mathbb{R}^d)$. Show that
   $$\|uvw\|_{L^1} \leq \|u\|_{L^{p_1}}\|v\|_{L^{p_2}}\|w\|_{L^{p_3}}.$$
   b) Show that $|f(x-y)||g(y)| = |f(x-y)|^{p/r}|g(y)|^{q/r}|f(x-y)|^{1-p/r}|g(y)|^{1-q/r}$.
   c) Conclude.

Problem 13. Let p and q in $[1,\infty]$ such that $p < q$. Show that if $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, then $f \in L^r(\mathbb{R}^d)$ for every $r \in [p,q]$.
Hint: Use that if $r \in [p,q]$, then there exists $\theta \in [0,1]$ such that $1/r = \theta/p + (1-\theta)/q$ and show that $\|f\|_{L^r} \leq \|f\|_{L^p}^{\theta}\|f\|_{L^q}^{1-\theta}$.

Problem 14. Let f, g and h in $\mathscr{S}(\mathbb{R}^d)$. Show the following properties

- $f \star g = g \star f$,
- $f \star (g+h) = f \star g + f \star h$,
- $(f \star g) \star h = f \star (g \star h)$.

Problem 15. Let us consider a real valued function $u : [0,+\infty[\times\mathbb{R}^d \to \mathbb{R}$. solution of the wave equation.
$$\partial_t^2 u - \Delta u = 0 \quad \text{in }]0,+\infty[\times\mathbb{R}^d.$$

Assume that

(H1) $u \in C_b^2([0,+\infty[\times\mathbb{R}^d)$;

(H2) there exists $R > 0$, such that $u(0,\cdot)$ and $\partial_t u(0,\cdot)$ vanish on $B(0,R) := \{x \in \mathbb{R}^d \mid |x| \leq R\}$.

The goal of this exercise is to show that
$$u = 0 \ \text{ in } K(R) := \{(t,x) \in [0,+\infty[\times\mathbb{R}^d \mid |x| \leq R - t\}.$$

Part 1

For any $\varepsilon \geq 0$ and $(t,x) \in [0,+\infty[\times\mathbb{R}^d$, we set

$$\varphi_\varepsilon(t,x) := R - (t + \sqrt{|x|^2+\varepsilon}).$$

1. Sow that for any $t \in [0,+\infty[$ and $s > 0$, the following quantity

$$E_s^\varepsilon(t) := \frac{1}{2}\int_{\mathbb{R}^d} e^{2s\varphi_\varepsilon(t,x)}(|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2)dx,$$

   is well-defined.

2. Assume that $\varepsilon > 0$.
   a) Show that

$$\frac{d}{dt}E_s^\varepsilon = -s\int_{\mathbb{R}^d} e^{2s\varphi_\varepsilon}(|\partial_t u|^2 + |\nabla u|^2)dx - 2s\int_{\mathbb{R}^d} e^{2s\varphi_\varepsilon}(\nabla\varphi_\varepsilon \cdot \nabla u)\partial_t u dx.$$

   b) Show that $\|\nabla\varphi_\varepsilon(t,\cdot)\|_{L^\infty} \leq 1$.
   (*Hint*: recall that $\|\nabla\varphi_\varepsilon(t,\cdot)\|_{L^\infty} = \sup_{x\in\mathbb{R}^d}\left(\sum_{j=1}^d |\partial_j\varphi_\varepsilon(t,x)|^2\right)^{1/2}$).
   c) Show that

$$-2\int_{\mathbb{R}^d} e^{2s\varphi_\varepsilon}(\nabla\varphi_\varepsilon \cdot \nabla u)\partial_t u dx \leq \int_{\mathbb{R}^d} e^{2s\varphi_\varepsilon}(|\partial_t u|^2 + |\nabla u|^2)dx.$$

   (*Hint*: use the estimate $2ab \leq a^2 + b^2$)
   d) Deduce that

$$\forall t \in [0,+\infty[, \quad E_s^\varepsilon(t) \leq E_s^\varepsilon(0).$$

3. Deduce from the dominated convergence theorem that

$$\forall t \in [0,+\infty[, \quad E_s^0(t) \leq E_s^0(0).$$

4. Deduce from 3. that

$$\forall t \in [0,+\infty[, \quad \lim_{s\to+\infty} E_s^0(t) = 0.$$

   (*Hint*: use that $\varphi_0(0,x) < 0$ when $x \in B(0,R)$ and (**H2**)).

5. Conclude that

$$\forall (t,x) \in K(R), \quad u(t,x) = 0.$$

Problem 16. (Heat equation in L^p (I)) For any $t > 0$ and $f \in \mathscr{S}(\mathbb{R}^d)$, we define the function $e^{t\Delta}f$ by

$$e^{t\Delta}f := f \star h_t,$$

where

$$\forall y \in \mathbb{R}^d, \quad h_t(y) := \frac{1}{(4\pi t)^{\frac{d}{2}}}e^{-\frac{|y|^2}{4t}}.$$

1. Let $f \in \mathscr{S}(\mathbb{R}^d)$. What is the Cauchy problem satisfied by $u : (t,x) \in]0,+\infty[\times\mathbb{R}^d \mapsto e^{t\Delta} f \in \mathbb{R}$.

2. Let $p \in [1,\infty[$ and $t > 0$. Show that for any $q \in [p,\infty[$, $e^{t\Delta}$ extends to a continuous operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ and that
$$\|e^{t\Delta}\|_{\mathrm{B}(L^p,L^q)} \leq \|h_t\|_{L^{(1+1/q-1/p)^{-1}}}.$$

3. Show that for any $p \in [1,\infty[$, $q \in [p,\infty[$ and $t > 0$, we have
$$\|e^{t\Delta}\|_{\mathrm{B}(L^p,L^q)} \leq \frac{1}{t^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}}.$$

4. Show that for any $f \in L^p(\mathbb{R}^d)$ with $p \in [1,\infty[$, the function $u : (t,x) \in]0,+\infty[\times\mathbb{R}^d \mapsto e^{t\Delta} f \in \mathbb{R}$ belong in $C^\infty(]0,+\infty[\times\mathbb{R}^d)$ and satisfies the heat equation.

5. Let $p \in [1,\infty[$ and $f \in L^p(\mathbb{R}^d)$. Show that $\lim_{t\to 0^+} e^{t\Delta} f = f$ in $L^p(\mathbb{R}^d)$.

Problem 17. (Schrödinger equation in L^2 (I)) For any $t \in R$ and $f \in \mathscr{S}(\mathbb{R}^d)$, we define the function
$$e^{it\Delta} f := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{it|\xi|^2} \widehat{f}(\xi) d\xi.$$

1. Show that for any $t \in \mathbb{R}$ the operator $e^{it\Delta}$ extends to an operator from $L^2(\mathbb{R}^d)$ into itself and that
$$\forall f \in L^2(\mathbb{R}^d), \quad \|e^{it\Delta} f\|_{L^2} = \|f\|_{L^2}.$$

Problem 18. (Schrödinger equation in L^2 (II)) Let t and s in $\mathbb{R}$. Show that

- $e^{i0\Delta} = \mathrm{Id}_{L^2}$.
- $e^{it\Delta} \circ e^{is\Delta} = e^{i(s+t)\Delta}$.
- $(e^{it\Delta})^* = e^{-it\Delta}$.

C. Notation

Symbol	Explanation	Page
$\mathbb{N}$	Natural numbers (not including zero!)	
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$	
D	Differential of a vector-valued function	
grad	Gradient of a scalar function, $\operatorname{grad} f = Df$	
div	Divergence of a vector field, $\operatorname{div} v = \operatorname{Tr}(Dv)$	
$B(x, r)$	Open ball of radius r around x	
$\mathscr{S}(\mathbb{R}^d)$	Space of Schwartz functions on $\mathbb{R}^d$	5
$\mathscr{S}'(\mathbb{R}^d)$	Space of tempered distributions on $\mathbb{R}^d$	17
$L^p(\mathbb{R}^d)$	Lebesgue space of p-integrable functions	25
$H^k(\mathbb{R}^d)$	Sobolev space of functions in $L^2(\mathbb{R}^d)$ with k weak derivatives in L^2	21
X	Usually a complex Banach space	
$\mathrm{B}(X, Y)$	Banach space of bounded linear operators from X to Y	
$\mathrm{B}(X)$	Banach space of bounded linear operators from X to X	
X'	Space of continuous linear functionals on X (=$\mathrm{B}(X, \mathbb{C})$)	17
$\mathcal{H}$	Complex (separable) Hilbert space	
$A, D(A)$	Densely defined linear operator	??
$\mathscr{G}(A)$	Graph of A	??
$\overline{A}$	Closure of $(A, D(A))$	??
$\lVert\cdot\rVert_{D(A)}$	Graph norm on $D(A)$	??
A^*	(Hilbert) adjoint of $(A, D(A))$	??
$\ker(A)$	Kernel of A	
$\operatorname{ran}(A)$	Range of A	
$\rho(A)$	Resolvent set of A	??
$R_z(A)$	Resolvent of A in $z \in \rho(A)$, $(A - z)^{-1}$	??
$\sigma(A)$	Spectrum of A	??
$C^k(U)$	Space of k-times continuously differentiable functions $U \to \mathbb{C}$	
$C_0^k(U)$	Space of k-times continuously differentiable functions $U \to \mathbb{C}$ with compact support, $\operatorname{supp} f \Subset U$	

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