Partial Differential Equations

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[1]These lecture notes are a draft and likely to contain mistakes. Please report any typos, errors, or suggestions to `jonas.lampart@u-bourgogne.fr`. Version of April 16, 2025

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1. Introduction

A partial differential equation (PDE) is an equation whose 'unknown' is a function u, and in which (partial) derivatives of that function appear. This is similar to an ordinary differential equation (ODE) but the difference is that the unknown function

$$u : \mathbb{R}^d \to \mathbb{R} \text{ (or } \mathbb{C})$$

depends on more than one variable, $d \geq 2$, and derivatives in different directions play a role. Such equations, or systems of equations, arise in many contexts mathematics and applications in physics, engineering, and the sciences – such as electrodynamics, quantum mechanics, dynamics of weather and climate, and the description of materials.

1.1. Examples

1. The heat equation
$$\partial_t u(t,x) = \Delta_x u(t,x) \tag{1.1}$$
describes diffusion of heat in a (homogeneous, isotropic) medium.

2. Schrödinger's equation
$$\mathrm{i}\partial_t \psi(t,x) = -\Delta_x \psi(t,x) + V(x)\psi(t,x) \tag{1.2}$$
describes the wave-function of a quantum particle in an external potential V.

3. The Poisson equation
$$\Delta u(x) = \rho(x) \tag{1.3}$$
gives the electric potential generated by the (static) charge distribution ρ. Maxwell's equations give a more complete description of electrodynamics.

4. The Euler equation
$$\begin{cases} \partial_t v(t,x) + v(t,x) \cdot D_x v(t,x) + \operatorname{grad}_x p(t,x) = 0 \\ \hfill \operatorname{div}_x v(t,x) = 0 \end{cases} \tag{1.4}$$
describes the velocity field $v : \mathbb{R}^d \to \mathbb{R}^d$ and pressure $p : \mathbb{R}^d \to \mathbb{R}$ of an incompressible, inviscid fluid. Similar systems, like the Navier-Stokes equations, are used to model the dynamics of fluids and gases with different properties, e.g. water waves or atmospheric currents.

5. The Cauchy-Riemann equations

$$\begin{cases} \partial_x u(x,y) - \partial_y v(x,y) = 0 \\ \partial_y u(x,y) + \partial_x v(x,y) = 0 \end{cases} \tag{1.5}$$

are satisfied by the real and imaginary part of every holomorphic function $f = u + \mathrm{i}v : \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{C}$.

Let $\alpha \in \mathbb{N}_0^d$ be a 'multi-index' and set

$$\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}}, \tag{1.6}$$

where $|\alpha| = \sum_{j=1}^d \alpha_j$. That is, α_j is the number of partial derivatives in direction j and $|\alpha|$ is the total number of derivatives. Since for $u \in C^k(U, \mathbb{C}^n)$ the partial derivatives can be taken in any order, we can thus express the tensor $D^k u$ by

$$(D^k u)_{j_1,\dots,j_k} = \frac{\partial^k u}{\partial x_{j_k} \cdots \partial x_{j_1}} = \partial^\alpha u \tag{1.7}$$

where α_i is the number of partial derivatives taken in the i-th direction, and $|\alpha| = k$.

Note that we have the generalised Leibniz rule

$$\partial^\alpha (fg) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g), \tag{1.8}$$

where $\beta \le \alpha$ if $\beta \le \alpha$ if $\beta_j \le \alpha_j$ for all $j = 1, \dots, d$, and the binomial coefficients are generalised as

$$\binom{\alpha}{\beta} = \prod_{j=1}^d \binom{\alpha_j}{\beta_j}. \tag{1.9}$$

Definition 1.1 (Linear PDE)**.** A PDE is called (inhomogneous) linear PDE of order k if it has the form

$$\sum_{|\alpha| \le k} a_\alpha(x) \partial^\alpha u = f(x), \tag{1.10}$$

where $a_\alpha : \mathbb{R}^d \to \mathbb{C}^{n \times n}$, for $|\alpha| \le k$, and $f : \mathbb{R}^d \to \mathbb{C}^n$. The functions a_α are called the coefficients, and the PDE is called homogeneous if $f = 0$.

Question 1.2. Which of the examples in Sect. 1.1 are linear (in-) homogeneous PDEs?

2. Linear PDEs with constant coefficients and the Fourier transform

A particularly simple case of linear differential equations are those with constant coefficients, where the functions $a_\alpha(x) \equiv a_\alpha$ are independent of x. These can be transformed into simpler equations by the Fourier transform.

For $f \in L^1(\mathbb{R}^d)$, the Fourier transform is defined by

$$\hat{f}(p) = (\mathscr{F} f)(p) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}p\cdot x} f(x)\mathrm{d}x. \tag{2.1}$$

Formally, we have with $p^\alpha = \prod_{j=1}^d p_j^{\alpha_j}$

$$\begin{aligned} p^\alpha \hat{f}(p) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} p^\alpha \mathrm{e}^{-\mathrm{i}p\cdot x} f(x)\mathrm{d}x \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (-\mathrm{i})^{-|\alpha|} (\partial_x^\alpha \mathrm{e}^{-\mathrm{i}p\cdot x}) f(x)\mathrm{d}x \\ &\overset{!}{=} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (\mathrm{i})^{|\alpha|} (-1)^{|\alpha|} \mathrm{e}^{-\mathrm{i}p\cdot x} \partial_x^\alpha f(x)\mathrm{d}x \\ &= (-\mathrm{i})^{|\alpha|} \widehat{\partial_x^\alpha f}(p), \end{aligned}$$

but the integration by parts (without boundary terms!) in the penultimate step certainly needs justification.

If we accept this identity, the linear PDE of Def. 1.1 becomes after transformation

$$\Big(\sum_{|\alpha|\le k} a_\alpha (\mathrm{i}p)^\alpha \Big) \hat{u}(p) = \hat{f}(p). \tag{2.2}$$

Any solution then satisfies, formally,

$$\hat{u}(p) \overset{!}{=} \Big(\sum_{|\alpha|\le k} a_\alpha (\mathrm{i}p)^\alpha \Big)^{-1} \hat{f}(p).$$

To recover the solution u, however, we will need to invert the Fourier transform.

2.1. Basic properties

A few important properties of the Fourier transform of $f \in L^1(\mathbb{R}^d)$ follow immediately from the definition.

Proposition 2.1. *Let $f \in L^1(\mathbb{R}^d)$, denote by $\hat{f}$ its Fourier transform* (2.1) *and denote by $Rf(x) = f(-x)$ the reflection at $x = 0$ and $T_a f(x) = f(x-a)$ the translation. Then*

a) $\widehat{T_a f} = \mathrm{e}^{-\mathrm{i}ap} \hat{f}$

b) $T_a \hat{f} = \widehat{\mathrm{e}^{\mathrm{i}ax} f}$

c) $R\hat{f} = \widehat{Rf}$

d) $\overline{\hat{f}} = R\hat{\overline{f}}$

e) If f is real and even ($Rf = f$) then $\hat{f}$ is also real and even.

Proof. Properties *a)–d)* follow from simple changes of variables (exercise). Property *e)* follows by combining *c*) and *d*). □

The Dominated Convergence Theorem A.9 also yields that $\hat{f}$ is continuous.

Lemma 2.2. *Let $f \in L^1(\mathbb{R}^d)$ and $\hat{f}$ its Fourier transform* (2.1)*, then $\hat{f}$ is continuous.*

Proof. Let $p_n \to p$ be a convergent sequence. Then since $|\mathrm{e}^{-\mathrm{i}p_n x} f(x)| \leq |f(x)| \in L^1(\mathbb{R}^d)$

$$\lim_{n\to\infty} \hat{f}(p_n) = \lim_{n\to\infty} \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{-\mathrm{i}p_n x} f(x) \mathrm{d}x = \frac{1}{(2\pi)^{d/2}} \int \lim_{n\to\infty} \mathrm{e}^{-\mathrm{i}p_n x} f(x) \mathrm{d}x = \hat{f}(p) \tag{2.3}$$

by Dominated Convergence A.9, which proves the claim. □

2.2. The Schwartz space $\mathscr{S}$

In order to make the formal calculations from the introduction rigorous and derive consequences for the solutions to the PDE, we start by introducing a class of functions on which the calculations can easily be justified. We will later expand beyond this class by approximation arguments.

A good framework to consider identities such as (2.2) is the space of Schwartz functions, where we can

- differentiate
- multiply by polynomials
- define the Fourier transform and its inverse.

Definition 2.3. The Schwartz space is

$$\mathscr{S}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) \middle| \forall \alpha, \beta \in \mathbb{N}_0^d : \sup_{x \in \mathbb{R}^d} |x^\alpha \partial_x^\beta f(x)| < \infty \right\}. \tag{2.4}$$

A sequence f_n, $n \in \mathbb{N}$ in $\mathscr{S}$ converges to $f \in \mathscr{S}$ iff

$$\forall \alpha, \beta \in \mathbb{N}_0^d : \lim_{n\to\infty} \|f_n - f\|_{\alpha,\beta} = 0, \tag{2.5}$$

where

$$\|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} |x^\alpha \partial_x^\beta f(x)|. \tag{2.6}$$

A map $T : \mathscr{S}(\mathbb{R}^d) \to X$ into a metric space X is continuous iff T is sequentially continuous, that is, if for every sequence f_n converging to $f \in \mathscr{S}(\mathbb{R}^d)$

$$\lim_{n\to\infty} Tf_n = Tf \tag{2.7}$$

converges in X.

Question 2.4. Which of the following functions are elements of $\mathscr{S}(\mathbb{R})$?

1. $x \mapsto \cos(x)$,
2. $x \mapsto \cosh(x)^{-1} = 2(e^x + e^{-x})^{-1}$,
3. $x \mapsto e^{-|x|}$,
4. $x \mapsto e^{-x^2}$.

Remark 2.5. The space $\mathscr{S}$ is a complete metric space with the distance

$$d(f,g) = \sum_{n\in\mathbb{N}_0} 2^{-n} \max_{|\alpha|+|\beta|=n} \frac{\|f-g\|_{\alpha,\beta}}{1+\|f-g\|_{\alpha,\beta}}. \tag{2.8}$$

The notion of convergence defined above is the same as the convergence in the metric d.

Remark 2.6. Functions in $\mathscr{S}$ are smooth by definition, and decrease faster than any inverse polynomial. Hence $\mathscr{S} \subset L^\infty$ with $\|f\|_\infty = \|f\|_{0,0}$, and $\mathscr{S} \subset L^p$ for any $1 \le p < \infty$, as by the multinomial formula

$$\begin{aligned} |f(x)| &\le (1+x^{2d})^{-1} \sup_{y\in\mathbb{R}^d} |(1+y^{2d})f(y)| \\ &\le (1+x^{2d})^{-1} \sup_{y\in\mathbb{R}^d} \Big|\Big(1+\sum_{|\alpha|=d} \frac{d!}{\alpha!} y^{2\alpha}\Big) f(y)\Big| \\ &\le (1+x^{2d})^{-1} (\|f\|_{0,0} + \sum_{|\alpha|=d} \frac{d!}{\alpha!} \|f\|_{2\alpha,0}), \end{aligned}$$

and

$$\int (1+x^{2d})^{-1} < \infty \tag{2.9}$$

for $p \ge 1$.

For $f, g \in \mathscr{S}(\mathbb{R}^d)$ we define the convolution by

$$(f*g)(x) := \int f(x-y)g(y)\mathrm{d}y. \tag{2.10}$$

Lemma 2.7. *Let $f, g \in \mathscr{S}(\mathbb{R}^d)$ with $\int g = 1$ and set $g_n(x) = n^d g(nx)$, then*

$$\lim_{n\to\infty} (f * g_n)(x) = f(x).$$

Proof. By a change of variable

$$\int f(x-y) n^d g(ny) \mathrm{d}y = \int f(x - n^{-1}y) g(y) \mathrm{d}y. \tag{2.11}$$

Now the integrand converges pointwise to $f(x)g(y)$ and is bounded by $\|f\|_\infty |g(y)| \in L^1$, so the integral converges to $f(x) \int g = f(x)$ by Dominated Convergence. □

With this Lemma, we can prove the Fourier inversion theorem on $\mathscr{S}$.

Proposition 2.8. *Define*

$$(\mathscr{F}^{-1} f)(x) := \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{\mathrm{i}px} f(p) \mathrm{d}p.$$

Then for all $f \in \mathscr{S}(\mathbb{R}^d)$,

$$f = \mathscr{F}^{-1} \mathscr{F} f = \mathscr{F} \mathscr{F}^{-1} f.$$

Proof. We admit that $\hat{f} \in \mathscr{S}$, which is proved in Proposition 2.9 below. Let $g(x) = \mathrm{e}^{-x^2/2}$ and $g_n(x) = g(n^{-1}x)$. Then

$$(\mathscr{F}^{-1} \hat{f}(x) = \lim_{n\to\infty} \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{\mathrm{i}px} g_n(p) f(p) \mathrm{d}p \tag{2.12}$$

by Dominated Convergence. On the other hand, by Fubini,

$$\frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{\mathrm{i}px} g_n(p) \hat{f}(p) \mathrm{d}p = \frac{1}{(2\pi)^{d}} \int \mathrm{e}^{\mathrm{i}px} \mathrm{e}^{-\mathrm{i}py} g_n(p) f(y) \mathrm{d}y \mathrm{d}p = \frac{(\hat{g}_n * f)(x)}{(2\pi)^{d/2}}. \tag{2.13}$$

Now $\hat{g}_n(x) = n^d \mathrm{e}^{-n^2 x^2/2}$ (see Problem 2), so by the preceding lemma

$$\lim_{n\to\infty} (\hat{g}_n * f)(x) = f(x) \int g = (2\pi)^{d/2} f(x), \tag{2.14}$$

and thus $(\mathscr{F}^{-1}\hat{f})(x) = f(x)$. The proof for $\mathscr{F}(\mathscr{F}^{-1} f)(x) = f(x)$ is the same. □

Proposition 2.9. *The Fourier transform $\mathscr{F}$ is a linear and continuous map*

$$\mathscr{F} : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d), \qquad f \mapsto \hat{f}.$$

Its continuous inverse is given by $\mathscr{F}^{-1}$. Moreover, the identities

$$(\partial^\alpha \mathscr{F} f)(p) = (\mathscr{F} (-\mathrm{i}x)^\alpha f)(p) \tag{2.15}$$

$$p^\alpha (\mathscr{F} f)(p) = \mathscr{F}\big((-\mathrm{i})^{|\alpha|} \partial^\alpha) f\big)(p) \tag{2.16}$$

hold for all $f \in \mathscr{S}(\mathbb{R}^d)$, $\alpha \in \mathbb{N}_0^d$.

Proof. We prove that $\hat{f}$ is smooth and the first identity by induction on $|\alpha|$. For $|\alpha| = 0$ we only need to prove that $\hat{f}$ is continuous, which is Lemma 2.2.

Now assume the statement holds for all $\beta \in \mathbb{N}_0^d$ with $|\beta| \le k$ and let $|\alpha| = k+1$. Then there are $\beta \in \mathbb{N}_0^d$ and $j \in \{1, \dots, d\}$ with $\alpha = \beta + e_j$. Denote $g = \partial^\beta \hat{f}$. By the theorem on parameter-dependent integrals A.10 and the induction hypothesis

$$g(p) = \frac{1}{(2\pi)^{d/2}} \int e^{-ipx} (-ix_j)^\beta f(x) dx \tag{2.17}$$

is differentiable, with

$$\partial_{p_j} g(p) = \frac{1}{(2\pi)^{d/2}} \int \partial_{p_j} e^{-ipx} (-ix_j)^\beta f(x) dx = \frac{1}{(2\pi)^{d/2}} \int e^{-ipx} (-ix_j)^\alpha f(x) dx. \tag{2.18}$$

This completes the induction.

For the second identity, we use that

$$\begin{aligned} p_j \hat{f}(p) &= \frac{1}{(2\pi)^{d/2}} \int p_j e^{-ipx} f(x) dx = \frac{1}{(2\pi)^{d/2}} \int i\partial_{x_j} e^{-ipx} f(x) dx \\ &= \frac{1}{(2\pi)^{d/2}} \int i\partial_{x_j} (e^{-ipx} f(x)) - ie^{-ipx} \partial_{x_j} f(x) dx. \end{aligned} \tag{2.19}$$

The integral of the derivative vanishes, because for $h \in \mathscr{S}$ by Fubini's Theorem A.6 and the fundamental theorem of calculus

$$\begin{aligned} \int \partial_{x_j} h(x) dx &= \int_{|x_j| \le R} \partial_{x_j} h(x) dx + \int_{|x_j| > R} \partial_{x_j} h(x) dx \\ &= \int_{\mathbb{R}^{d-1}} h(x) \Big|_{x_j=-R}^{x_j=R} + \int_{|x_j|>R} \partial_{x_j} h(x) dx, \end{aligned} \tag{2.20}$$

and

$$\lim_{R\to\infty} \int_{\mathbb{R}^{d-1}} h(x) \Big|_{x_j=-R}^{x_j=R} = 0 = \int_{|x_j|>R} \partial_{x_j} h(x) dx, \tag{2.21}$$

since h vanishes faster than any polynomial. This proves the second identity in the case $|\alpha| = 1$, from which the general case follows by induction, like the first.

We have shown that $\hat{f}$ is smooth, so to show that $\hat{f} \in \mathscr{S}$ we need to show that $\|\hat{f}\|_{\alpha,\beta}$ is finite. Using the identities, we find using the Leibniz rule (see Problem 4)

$$\begin{aligned} \|\hat{f}\|_{\alpha,\beta} &= \sup_{p\in\mathbb{R}^d} \left| (\mathscr{F} \partial^\alpha x^\beta)(p) f \right| \\ &\le \frac{1}{(2\pi)^{d/2}} \int (1 + x^{2d})^{-1} \sup_{y\in\mathbb{R}^d} (1 + y^{2d}) |\partial^\alpha y^\beta f(y)| \\ &\le C \sum_{\substack{|\gamma| \le \beta + 2d \\ |\delta| \le |\alpha|}} \|f\|_{\gamma,\delta} \end{aligned} \tag{2.22}$$

for some constant C, and hence $\hat{f} \in \mathscr{S}$. Moreover, $\mathscr{F}$ is continuous in $f = 0$ by the bound (2.22), so it is continuous by linearity. Continuity of the inverse follows from $\mathscr{F}^{-1} = R\mathscr{F}$. □

Corollary 2.10. *Let $f, g \in \mathscr{S}(\mathbb{R}^d)$, then*

$$\int f(x)\hat{g}(x) = \int \hat{f}(x)g(x),$$

and

$$\int |\widehat{f}|^2(p)\mathrm{d}p = \int |f|^2(x)\mathrm{d}x.$$

Proof. The first statement follows directly from Fubini's theorem. The second is a consequence of this and the Fourier inversion formula together with $\mathscr{F}^{-1}f(x) = \hat{f}(-x) = Rf$ and Proposition 2.1d), i.e.,

$$\int \hat{f}(p)\overline{\hat{f}}(p)\mathrm{d}p = \int f(x)\hat{\overline{\hat{f}}}(x)\mathrm{d}x = \int f(x)\widehat{R\hat{\overline{f}}}(x)\mathrm{d}x = \int |f(x)|^2\mathrm{d}x. \tag{2.23}$$

□

Example 2.11. Let $z \in \mathbb{C}$, $f \in \mathscr{S}(\mathbb{R}^d)$ and consider the linear PDE

$$(\Delta + z)u = f. \tag{2.24}$$

Assuming that $u \in \mathscr{S}$, we can take the Fourier transform and obtain

$$(-p^2 + z)\hat{u}(p) = \hat{f}(p). \tag{2.25}$$

If $z \in \mathbb{C} \setminus \mathbb{R}_+$, then $-p^2 + z \neq 0$, and

$$\hat{u}(p) = (-p^2 + z)^{-1}\hat{f}(p) \in \mathscr{S}. \tag{2.26}$$

In this case, the unique solution $u \in \mathscr{S}(\mathbb{R}^d)$ to (2.24) is given by

$$u(x) = \mathscr{F}^{-1}(-p^2 + z)^{-1}\hat{f}. \tag{2.27}$$

Uniqueness holds only with the requirement that $u \in \mathscr{S}$. Without this hypothesis, we can add any solution v of the homogeneous equation

$$(\Delta + z)v = 0, \tag{2.28}$$

for example $v_\pm = \mathrm{e}^{\pm\sqrt{-z}x}$ for $d = 1$, $z \neq 0$. Note that these solutions are not elements of $\mathscr{S}$, as they do not decay for $|x| \to \infty$!

If $z \in \mathbb{R}_+$ the situation is more complicated as $-p^2 + z$ is not smoothly invertible, but if $\hat{f}$ has the same zeros the solution might still be an element of $\mathscr{S}$.

Example 2.12. (The heat equation on $\mathscr{S}$) If we take the Fourier transform of the heat equation

$$\partial_t u = \Delta u \tag{2.29}$$

in both t and x, we obtain

$$(\mathrm{i}\tau + p^2)\mathscr{F}_{t,x}u = 0. \tag{2.30}$$

In the best case this would tell us that $u = 0$ (though this is not clear since the multiplier vanishes at $(\tau, p) = 0$). However, the equation is an evolution equation and $\mathscr{S}(\mathbb{R} \times \mathbb{R}^d)$ is not a natural space for the solutions. Indeed, $u \in \mathscr{S}(\mathbb{R} \times \mathbb{R}^d)$ would mean that $u(t, x) \to 0$ for $t \to \pm\infty$, but instead of this restriction we should rather specify initial data, as for ODEs.

If we only take the Fourier transform in x, we obtain

$$\partial_t \hat{u}(t, p) = -p^2 \hat{u}(t, p). \tag{2.31}$$

If we fix an initial condition $\hat{u}_0(p) = \hat{u}(0, p) \in \mathscr{S}(\mathbb{R}^d)$ the equation is an ODE initial value problem for every p. The unique solution is

$$\hat{u}(t, p) = \mathrm{e}^{-p^2 t} \hat{u}_0(p), \tag{2.32}$$

and for every $t \geq 0$ this is again an element of $\mathscr{S}(\mathbb{R}^d)$. Moreover, $\lim_{t \to 0} \mathrm{e}^{-p^2 t} \hat{u}_0(p) = u_0$ in $\mathscr{S}(\mathbb{R}^d)$.

With this we can see that there exists a unique function

$$(t, x) \mapsto u(t, x), \qquad u \in C^1((0, \infty) \times \mathbb{R}^d, \mathbb{C}), \qquad u(t, \cdot) \in \mathscr{S}(\mathbb{R}^d) \tag{2.33}$$

satisfying the heat equation (2.29) and such that

$$\lim_{t \to 0} u(t, \cdot) = u_0 \tag{2.34}$$

in $\mathscr{S}(\mathbb{R}^d)$.

2.3. Convolution and approximation

The convolution of functions appears frequently in formulas for solutions of PDEs. It is also an important technical tool that enables us to approximate functions with singularities by smooth functions.

For $f, g \in \mathscr{S}(\mathbb{R}^d)$, we define their convolution as

$$(f * g)(x) = \int f(x - y) g(y) \mathrm{d}y. \tag{2.35}$$

Note that the formula for $f * g$ remains well defined whenever $f(x - y)g(y)$ is integrable for every $x \in \mathbb{R}^d$. By the Hölder inequality, Theorem A.5, this holds if $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ with $p^{-1} + q^{-1} = 1$. In particular, if $g \in \mathscr{S}(\mathbb{R}^d)$, and thus in L^q for all $1 \leq q \leq \infty$, the $f * g$ is defined for $f \in L^p(\mathbb{R}^d)$ and all $1 \leq p \leq \infty$.

The convolution of f with $g \in \mathscr{S}$ has a smoothing effect, which we will use this to approximate arbitrary elements of $L^p(\mathbb{R}^d)$ by smooth functions.

Theorem 2.13. *Let $1 \leq p \leq \infty$, $f \in L^p(\mathbb{R}^d)$ and $g \in \mathscr{S}(\mathbb{R}^d)$. Then*

a) $f * g \in C^\infty(\mathbb{R}^d)$;

b) If $(1+x^2)^r f(x) \in L^p(\mathbb{R}^d)$ *for some* $r > 0$*, then* $(1+x^2)^r (f*g)(x) \in L^\infty(\mathbb{R}^d)$*;*

c) If $f \in \mathscr{S}(\mathbb{R}^d)$ *then*

$$\widehat{f*g} = (2\pi)^{d/2}\hat{f}\hat{g}, \qquad \widehat{fg} = (2\pi)^{-d/2}\hat{f}*\hat{g}.$$

Proof. a): Changing variables with $z = x - y$ we have

$$(f*g)(x) = \int f(z)g(x-z)\mathrm{d}z. \tag{2.36}$$

For given x_0, we consider a neighbourhood $U = B(x_0, 1)$. Since $g \in \mathscr{S}$, the derivatives of the integrand satisfy for $x \in U$

$$|\partial_x^\alpha f(z)g(x-z)| \le C|f|(z)(1+|x_0 - z|^{2d})^{-1} \tag{2.37}$$

with some $C > 0$. The right hand side is integrable, since $(1+|x_0 - z|^{2d})^{-1} \in L^q(\mathbb{R}^d)$ for all $1 \le q \le \infty$. Thus, $f*g$ is smooth by Theorem A.10

b): We write

$$(1+x^2)^r(f*g)(x) = \int (1+|x-y|^2)^r f(x-y)(1+y^2)^r g(y)\Big(\frac{1+x^2}{(1+|x-y|^2)(1+y^2)}\Big)^r \mathrm{d}y. \tag{2.38}$$

Now

$$\frac{1+x^2}{(1+|x-y|^2)(1+y^2)} \le 4 \tag{2.39}$$

since for $|x-y| \le |x|/2$ we have $|y| \ge |x|/2$, and thus

$$\frac{1+x^2}{(1+|x-y|^2)(1+y^2)} \le \frac{1+x^2}{1+x^2/4} \le 4, \tag{2.40}$$

and the same argument with reversed roles in the denominator for $|x-y| \ge |x|/2$. Thus, by Hölder's inequality (A.7),

$$\begin{aligned} &\Big| \int (1+|x-y|^2)^r f(x-y)(1+y^2)^r g(y)\Big(\frac{1+x^2}{(1+|x-y|^2)(1+y^2)}\Big)^r \Big| \\ &\qquad \le 4^r \Big(\int |(1+|x-y|^2)^r f(x-y)|^p \mathrm{d}y\Big)^{1/p} \Big(\int (1+y^2)^r |g(y)|^q(y)\mathrm{d}y\Big)^{1/q}, \end{aligned} \tag{2.41}$$

which proves the claim by a change of variables in the integral involving f.

c) By *b)*, $f*g \in L^1(\mathbb{R}^d)$, so the Fourier transform is well defined. Now

$$\widehat{f*g} = \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{-\mathrm{i}p(x-y)-\mathrm{i}py} f(x-y)g(y)\mathrm{d}y\mathrm{d}x = (2\pi)^{d/2}\hat{f}(p)\hat{g}(p). \tag{2.42}$$

To prove the second formula, apply this to $\check{f} = \mathscr{F}^{-1}f$, $\check{g} = \mathscr{F}^{-1}g$ and use the inversion formula to find

$$\widehat{fg}(p) = (2\pi)^{-d/2}\check{f}*\check{g}(-p) = (2\pi)^{-d/2}\int \hat{f}(p+q)\hat{g}(-q)\mathrm{d}q = (2\pi)^{-d/2}\hat{f}*\hat{g}. \tag{2.43}$$

$\square$

The important idea to keep in mind from this theorem is that $f * g$ inherits the regularity of the more regular of f, g and the decay of the more slowly decaying. In particular, $\mathscr{S}$ is invariant under convolution.

Corollary 2.14. *For $g \in \mathscr{S}(\mathbb{R}^d)$, the map $f \mapsto f * g$ maps $\mathscr{S}(\mathbb{R}^d)$ to itself and is continuous.*

Proof. The fact that $f * g \in \mathscr{S}(\mathbb{R}^d)$ follows from points *a), b)* of the previous theorem. Continuity follows from its proof, namely the bounds (2.37), (2.38) which show that $\|f * g\|_{\alpha,\beta}$ can be bounded in terms of $\|f\|_{\gamma,\beta}$ for finitely many γ. □

The convolution satisfies some important inequalities with respect to the norms on the Lebesgue spaces.

Lemma 2.15. *Let $g \in \mathscr{S}(\mathbb{R}^d)$ and $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, then*

$$\|f * g\|_p \leq \|f\|_p \|g\|_1. \tag{2.44}$$

Proof. Let $q = p/(1-p)$ and write $|g(y)| = |g(y)|^{1/p}|g(y)|^{1/q}$. Then, by Hölder's inequality,

$$\begin{aligned} \int \left| \int f(x-y)g(y)\mathrm{d}y \right|^p \mathrm{d}x &\leq \int \int |f(x-y)|^p |g(y)| \mathrm{d}y \mathrm{d}x \Big(\int |g(z)| \mathrm{d}z \Big)^{p/q} \\ &= \|f\|_p^p \|g\|_1^{1+p/q}. \end{aligned} \tag{2.45}$$

Taking the p-th root and using again $1/p + 1/q = 1$ yields the claim.

□

These bounds show that, for example, the map $f \mapsto f * g$ for $g \in \mathscr{S}(\mathbb{R}^d)$ is continuous from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$, since it is bounded.

We will now use the convolution to approximate arbitrary elements of $L^p(\mathbb{R}^d)$ by smooth functions and extend the Fourier transform and convolution maps. For this, we first need a Lemma concerning the action of translations on L^p.

Lemma 2.16. *Let $T_a f(x) = f(x-a)$ be the translation and $f \in L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$. Then*

$$\lim_{a \to 0} \|T_a f - f\|_p = 0.$$

Proof. We start by reducing the proof to the case $p = 1$. For this, note that

$$\lim_{R \to \infty} \|\chi_{B(0,R)} f - f\|_p = 0 \tag{2.46}$$

by Dominated Convergence, since $|\chi_{B(0,R)} f - f|^p \leq |f|^p \in L^1(\mathbb{R}^d)$. Moreover, denoting by $A_R = \{x \in \mathbb{R}^d : ||f(x)| \leq R\}$, we also have

$$\lim_{R \to \infty} \|\chi_{A_R} f - f\|_p = 0, \tag{2.47}$$

by the same reasoning. Hence, given $\varepsilon > 0$ we can find $R > 0$ and f_R with $\operatorname{supp} f_R \subset B(0,R)$ and $|f_R| \leq R$ so that $\|f_R - f\|_p < \varepsilon$. Since T_a is an isometry, we thus have

$$\|T_a f - f\|_p \leq 2\varepsilon + \|T_a f_R - f_R\|_p, \tag{2.48}$$

and

$$\int |f_R(x-a) - f_R(x)|^p \mathrm{d}x \leq 2^{p-1} R^{p-1} \int |f_R(x-a) - f_R(x)| \mathrm{d}x, \tag{2.49}$$

which is finite because f_R has compact support. It is thus sufficient to prove the claim for $p = 1$.

Here, we need to use the fact from integration theory that any element of $L^1(\mathbb{R}^d)$ can be approximated in $L^1(\mathbb{R}^d)$ by a finite linear combination of characteristic functions on disjoint, half open rectangles [LL, Thm.1.18], that is, for $\varepsilon > 0$ there is

$$F = \sum_{i=1}^{N} \alpha_i \chi_{A_i} \tag{2.50}$$

with $\|F - f\|_1 < \varepsilon$. For any characteristic function χ_A of a rectangle

$$A = \{x \in \mathbb{R}^d : c_j < x_j \leq d_j\}, \tag{2.51}$$

$|\chi(x-a) - \chi(x)|$ is the characteristic function of the symmetric difference of A and its translate, whose volume is smaller than $2|a||\partial A|$, where $|\partial A|$ is the surface volume of A. We thus have

$$\|T_a f - f\|_1 \leq 2\varepsilon + 2|a| \sum_{j=1}^{N} |\alpha_i||\partial A_i|, \tag{2.52}$$

which is less than 3ε if $|a|$ is small enough. This proves the claim. □

Theorem 2.17 (Approximation by smooth functions)**.** *Let $g \in \mathscr{S}(\mathbb{R}^d)$ with $\int g = 1$ and set $g_n = n^d g(nx)$. For $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^d)$, the functions $f_n := f * g_n$ are smooth, and*

$$\lim_{n\to\infty} \|f_n - f\|_p = 0.$$

Proof. Smoothness of f_n is Theorem 2.13a). In order to prove the approximation in L^p, we write

$$f_n(x) - f(x) = \int (f(x-y) - f(x)) g_n(y) \mathrm{d}y. \tag{2.53}$$

Let $\varepsilon > 0$. We start by by considering the restriction of the integral to $|y| < \delta$. We have (cf. (2.45)),

$$\begin{aligned} \int \left| \int_{|y|<\delta} (f(x-y) - f(x)) g_n(y) \mathrm{d}y \right|^p \mathrm{d}x &\leq \int \int_{|y|<\delta} (f(x-y) - f(x)|^p |g_n(y)| \mathrm{d}y \mathrm{d}x \|g\|_1^{p-1} \\ &\leq \|g\|_1^{p-1} \sup_{|y|\leq\delta} \int |f(x-y) - f(x)|^p \mathrm{d}x. \end{aligned} \tag{2.54}$$

By Lemma 2.16, there exists δ so that $\|T_y f - f\|_p \le \varepsilon$ for $|y| \le \delta$, and we choose δ in this way. Then, we can find n_0 so that for $n \ge n_0$

$$\int_{|y|>\delta} |g_n(y)| = \int_{|y|>n\delta} |g(y)| \le \varepsilon. \tag{2.55}$$

By the triangle inequality for the L^p norm, we thus have by Lemma 2.15

$$\|f_n - f\|_p \le \varepsilon + 2\|f\|_p \varepsilon \tag{2.56}$$

which proves the claim since ε was arbitrary. □

Corollary 2.18. *The Schwartz functions $\mathscr{S}(\mathbb{R}^d)$ are dense in $L^p(\mathbb{R}^d)$ for $1 \le p < \infty$.*

Proof. Given $f \in L^p$ and $\varepsilon > 0$ we need to find $f_\varepsilon \in \mathscr{S}$ with $\|f - f_\varepsilon\|_p < \varepsilon$. To do this, first choose R so that $\|f - \chi_{B(0,R)} f\|_p < \varepsilon/2$. Then, let $f_{n,R} = \chi_{B(0,R)} f * g_n$ be a smooth approximation as in Theorem 2.17. This function is also rapidly decaying by Theorem 2.13b), since $\chi_{B(0,R)} f$ has compact support. Iterating this argument shows $f_{n,R} \in \mathscr{S}(\mathbb{R}^d)$. Choosing n large enough so that $\|f_{n,R} - \chi_{B(0,R)} f\| < \varepsilon/2$ yields the result. □

Remark 2.19. The Schwartz functions are clearly *not* dense in L^∞, since for the constant function $f \equiv 1$, $\|f - g\|_\infty = 1$ for all $g \in \mathscr{S}$, as g tends to zero for $|x| \to \infty$.

The density of $\mathscr{S}(\mathbb{R}^d)$ allows us to prove some important results on the Fourier transform on $L^p(\mathbb{R}^d)$ by approximation.

Theorem 2.20 (Riemann-Lebesgue Lemma)**.** *The Fourier transform is a continuous map $\mathscr{F} : L^1(\mathbb{R}^d) \to C_\infty(\mathbb{R}^d)$, where $C_\infty(\mathbb{R}^d)$ denotes the space of continuous functions tending to zero at infinity with the topology of uniform convergence.*

Proof. Continuity of $\hat{f}$ was already proved in Lemma 2.2. The fact that $\mathscr{F} : L^1(\mathbb{R}^d) \to C(\mathbb{R}^d)$ is continuous is then simply the inequality

$$\|\hat{f}\|_\infty \le (2\pi)^{-d/2} \|f\|_1. \tag{2.57}$$

To see that $\hat{f}$ tends to zero, let $\varepsilon > 0$ and take $f_\varepsilon \in \mathscr{S}(\mathbb{R}^d)$ so that $\|f - f_\varepsilon\| \le \varepsilon$. Then

$$|\hat{f}(x)| \le (2\pi)^{-d/2} \varepsilon + \hat{f}_\varepsilon(x). \tag{2.58}$$

Since $\hat{f}_\varepsilon \in \mathscr{S}(\mathbb{R}^d)$ by Proposition 2.9, it tends to zero, so there is $R > 0$ such that $|\hat{f}_\varepsilon(x)| \le \varepsilon$ for $|x| < R$, and thus $|\hat{f}(x)| \le \varepsilon((2\pi)^{-d/2} + 1)$ for $|x| > R$. □

We can extend some of the maps we have defined of $\mathscr{S}$ to L^p by continuity, using the following.

Theorem 2.21 (B.L.T. Theorem)**.** *Let X, Y be Banach spaces and $D \subset X$ a dense subspace. Suppose $A : D \to Y$ is a bounded linear transformation, then there exists a unique bounded linear transformation $\overline{A} : X \to Y$ that extends A, and $\|\overline{A}\| = \|A\|$ holds.*

Proof. By [FA, Prop.1.7.2], A is continuous so the idea is to extend in such a way that preserves continuity.

Since $\overline{D} = X$, every $x \in X \setminus D$ is a limit point of of D, i.e. there exist $x_n \in D$, $n \in \mathbb{N}$, so that $x_n \to x$ as $n \to \infty$. The sequence x_n is Cauchy in X, and because A is bounded, we have

$$\|Ax_n - Ax_m\|_Y \le \|A\| \|x_n - x_m\|_X, \tag{2.59}$$

so the sequence Ax_n is Cauchy in Y. Since Y is complete, it thus converges to a limit $y \in Y$. We set

$$\overline{A}x := y. \tag{2.60}$$

This is well defined, for if $\tilde{x}_n \to x$ is another sequence, then $\tilde{x}_n - x_n \to 0$ and thus

$$\lim_{n\to\infty} A\tilde{x}_n = y + \lim_{n\to\infty} A(\tilde{x}_n - x_n) = y, \tag{2.61}$$

by continuity of A. Linearity of $\overline{A}$ follows from linearity of A and the limit. This extension is unique, for if $\tilde{A}$ were another bounded extension, it would be continuous by [FA, Prop.1.7.2] and $\tilde{A}x = y = \overline{A}x$ follows.

Moreover, we have by continuity of the norm

$$\|\overline{A}x\|_Y = \|\lim_{n\to\infty} Ax_n\|_Y = \lim_{n\to\infty} \|Ax_n\|_Y \le \|A\| \lim_{n\to\infty} \|x_n\|_X = \|A\| \|x\|_X, \tag{2.62}$$

so $\|\overline{A}\| \le \|A\|$. We also have $\|A\| \le \|\overline{A}\|$, since in one case the supremum is over D and in the other over X, which is larger. Thus $\overline{A}$ is bounded with $\|\overline{A}\| = \|A\|$. □

We can now define the Fourier transform on $L^2(\mathbb{R}^d)$, where it is a unitary map.

Theorem 2.22 (Fourier-Plancherel)**.** *The Fourier transform $\mathscr{F} : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ admits a unique continuous extension*

$$\overline{\mathscr{F}} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d).$$

This map is unitary, i.e.,

$$\langle f, \overline{\mathscr{F}} g\rangle = \langle \overline{\mathscr{F}} f, g\rangle, \qquad \|f\|_2 = \|\overline{\mathscr{F}} f\|_2$$

for all $f, g \in L^2(\mathbb{R}^d)$ and $\overline{\mathscr{F}}^* = \overline{\mathscr{F}}^{-1} = \overline{\mathscr{F}^{-1}}$.

Proof. By Corollary 2.10, $\mathscr{F} : \mathscr{S}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defines a bounded linear map with norm one. Since $\mathscr{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, the unique continuous extension of this map $\overline{\mathscr{F}} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ also has norm one. The identities of Corollary 2.10 carry over to the extension by continuity of the scalar product and imply unitarity of the extension. □

Remark 2.23. The map $\overline{\mathscr{F}}$ is not given by the integral formula (2.1), which does not make sense for an arbitrary element of $L^2(\mathbb{R}^d)$. However, for $f \in L^1 \cap L^2(\mathbb{R}^d)$ this formula holds (by uniqueness of the extension). Then for any sequence $(f_n) \subset L^1 \cap L^2(\mathbb{R}^d)$ with $f_n \to f$ in $L^2(\mathbb{R}^d)$,

$$\overline{\mathscr{F}} f(x) = \lim_{n\to\infty} \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{-\mathrm{i}px} f_n(x)\mathrm{d}x. \tag{2.63}$$

For example, taking $f_n = \chi_{B(0,n)} f$,

$$\overline{\mathscr{F}} f(x) = \lim_{n\to\infty} \frac{1}{(2\pi)^{d/2}} \int_{|x|\leq n} \mathrm{e}^{-\mathrm{i}px} f(x)\mathrm{d}x. \tag{2.64}$$

Remark 2.24. The Fourier transform can be extended continuously to

$$\overline{\mathscr{F}} : L^p(\mathbb{R}^d) \to L^{\frac{p}{p-1}}(\mathbb{R}^d) \tag{2.65}$$

for $1 \leq p \leq 2$. However, this is in general not surjective, as can be seen from the case $p = 1$, where the range is contained in (but not equal to) $C_\infty(\mathbb{R}^d)$.

For the convolution map, the extension yields:

Corollary 2.25. *For $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^d)$ the convolution map*

$$\mathscr{S}(\mathbb{R}^d) \to L^p(\mathbb{R}^d), \qquad g \mapsto f * g$$

has a unique continuous extension to $L^1(\mathbb{R}^d)$ with norm $\|f\|_p$. The map

$$L^p(\mathbb{R}^d) \times L^1(\mathbb{R}^d) \to L^p(\mathbb{R}^d), \qquad (f, g) \mapsto f * g$$

is bilinear and continuous in each argument.

Proof. This follows directly from the bound (2.44) and the BLT Theorem. □

2.4. Duality and tempered distributions

We have now extended the Fourier transform to L^2, but we cannot use this for solving PDEs yet, since we most functions in L^2 are not differentiable in the standard sense. In order to overcome this, we will extend all the operations from $\mathscr{S}$ to a much larger space $\mathscr{S}'$ called the tempered distributions. This space also contains L^2, but not all of its elements can be thought of as functions.

Definition 2.26. Let X be a topological vector space. Then we define the topological dual of X by

$$X' := \{\varphi : X \to \mathbb{C},\ \varphi \text{ is linear and continuous}\}.$$

Example 2.27. If X is a Banach (i.e., normed, complete) space, then $X' = \mathrm{B}(X, \mathbb{C})$ is also a Banach space, with the norm

$$\|\varphi\|_{X'} = \sup_{0\neq x \in X} \frac{|\varphi(x)|}{\|x\|}. \tag{2.66}$$

Example 2.28. If $X = \mathcal{H}$ is a Hilbert space, then $\mathcal{H}'$ and be indetified with $\mathcal{H}$ via the (anti-linear) isomorphism (cf. [FA, Thm.2.4.1])

$$\Phi : \mathcal{H} \to \mathcal{H}', \qquad \Phi(f)(g) \mapsto \langle f, g\rangle_{\mathcal{H}}. \tag{2.67}$$

Definition 2.29. Let X be a topological vector space. The weak topology on X' is the smallest topology so that for every $x \in X$ the evaluation

$$\iota_x : X' \to \mathbb{C}, \qquad \iota_x\varphi = \varphi(x) \tag{2.68}$$

is continuous. A sequence $(\varphi_n)_{n\in\mathbb{N}} \subset X'$ converges weakly to φ if

$$\forall x \in X : \lim_{n\to\infty} \varphi_n(x) = \varphi(x). \tag{2.69}$$

Definition 2.30. The space of tempered distributions is $\mathscr{S}'(\mathbb{R}^d) := (\mathscr{S}(\mathbb{R}^d))'$ equipped with the weak topology.

Remark 2.31. Since $\varphi \in \mathscr{S}'(\mathbb{R}^d)$ is linear, continuity is equivalent to continuity in $f = 0$, since if $f_n \to f$ in $\mathscr{S}$, then $f_n - f \to 0$, and

$$\lim_{n\to\infty} \varphi(f_n) = \varphi(f) \Leftrightarrow \lim_{n\to\infty} |\varphi(f_n) - \varphi(f)| = \lim_{n\to\infty} |\varphi(f_n - f)| = 0. \tag{2.70}$$

Question 2.32. Which of the following formulas define a tempered distribution on $\mathbb{R}$?

1. $f \mapsto f'(0)$,
2. $f \mapsto \int f^2(x)\mathrm{d}x$,
3. $f \mapsto \int e^{\sqrt{1+x^2}} f(x)\mathrm{d}x$,
4. $f \mapsto \int |x| f(x)\mathrm{d}x$.

Example 2.33. Let $g \in L^p(\mathbb{R}^d)$, $p \leq \infty$ then

$$f \mapsto \varphi_g(f) = \int_{\mathbb{R}^d} \overline{g}(x) f(x)\mathrm{d}x \tag{2.71}$$

defines an element of $\mathscr{S}'(\mathbb{R}^d)$. It is clearly linear, and if $f_n \to 0$ in $\mathscr{S}$, then by Hölder's inequality

$$|\varphi_g(f_n)| \leq C\|g\|_p \sum_{|\alpha|\leq 2d} \|f_n\|_{\alpha,0} \to 0, \tag{2.72}$$

so φ_g is continuous.

Many other classes of functions can be identified with tempered distributions by this formula.

Definition 2.34. A distribution $\varphi \in \mathscr{S}'(\mathbb{R}^d)$ is called a *regular distribution* if there exists $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ (i.e., $\chi_{B(0,R)} g \in L^1$ for all $R > 0$) such that $\varphi = \varphi_g$, that is

$$\forall f \in \mathscr{S}(\mathbb{R}^d) : \varphi(f) = \int_{\mathbb{R}^d} \overline{g}(x) f(x)\mathrm{d}x. \tag{2.73}$$

Proposition 2.35. *Let $\varphi = \varphi_g$ be a regular distribution, then g is unique. That is, if $h \in L^1_{\text{loc}}(\mathbb{R}^d)$ is such that $\varphi = \varphi_h$, then $h = g$ almost everywhere.*

Proof. We have to show that

$$\forall \in \mathscr{S}(\mathbb{R}^d) : \int \overline{g}(x) f(x) \mathrm{d}x = \int \overline{h}(x) f(x) \mathrm{d}x \implies g = h \text{ a.e..} \tag{2.74}$$

By additivity in in g, h we may consider $\eta = g - h$ and show that $\varphi_\eta = 0$ implies $\eta = 0$ a.e.. By choosing f of compact support, we may assume that $\eta \in L^1$, without loss of generality. Now let $g_n(x) = n^d \mathrm{e}^{-n^2 x^2/2} \in \mathscr{S}(\mathbb{R}^d)$. By hypothesis, for every $x \in \mathbb{R}^d$

$$g_n * \overline{\eta}(x) = \int \overline{\eta}(y) g_n(x - y) \mathrm{d}y = \varphi_\eta(g_n(x - \cdot)) = 0. \tag{2.75}$$

On the other hand, by Theorem 2.17, $g_n * \overline{\eta}$ converges to $\overline{\eta}$ in L^1, so $\eta = 0$ in L^1 and thus almost everywhere. □

We can extend many (linear) operations on $\mathscr{S}$ to $\mathscr{S}'$ by duality, i.e. taking the transpose.

Proposition 2.36. *Let X, Y be topological vector spaces and $T : X \to Y$ a linear, continuous map. Then there exists a unique weakly continuous map $T' : Y' \to X'$ satisfying*

$$(T'\varphi)(x) = \varphi(Tx).$$

Proof. The formula defines a unique map since $\varphi \in X'$ is completely determined by its evaluations. This map is linear, since

$$T'(a\varphi + \psi)(x) = (a\varphi + \psi)(Tx) = a\varphi(Tx) + \psi(Tx) = aT'\varphi(x) + T'\psi(x). \tag{2.76}$$

It is weakly (sequentially) continuous since

$$\lim_{n\to\infty} T'\varphi_n(x) = \lim_{n\to\infty} \varphi_n(Tx) = \varphi(Tx) = T'\varphi(x) \tag{2.77}$$

for any weakly convergent sequence $\varphi_n \overset{w}{\to} \varphi$. □

Examples 2.37.

a) Fourier transform $\mathscr{F}$. For $g \in \mathscr{S}(\mathbb{R}^d)$ we have

$$((\mathscr{F}^{-1})'\varphi_g)(f) = \varphi_g(\mathscr{F}^{-1}f) = \int \overline{g}(x)(\mathscr{F}^{-1}f)(x)\mathrm{d}x \overset{\text{Parseval}}{=} \int \overline{\hat{g}(p)} f(p) \mathrm{d}p = \varphi_{\hat{g}}(f), \tag{2.78}$$

so the action of $(\mathscr{F}^{-1})'$ on $\mathscr{S}'$ extends the one of $\mathscr{F}$ on $\mathscr{S}$. We will also denote this by

$$(\mathscr{F}^{-1})'\varphi = \mathscr{F}\varphi =: \hat{\varphi}. \tag{2.79}$$

b) Derivative: For any $\alpha \in \mathbb{N}^d$ we have $(\partial^\alpha)' : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ linear and continuous. In this way we can define derivatives of all tempered distributions, in particular all L^2-functions.

c) Multiplication by a monomial: In this case we have $(x^\alpha)'\varphi_g = \varphi_{x^\alpha g} =: x^\alpha \varphi_g$.

d) Convolution with a Schwartz function. For fixed $g \in \mathscr{S}(\mathbb{R}^d)$, the map

$$f \mapsto g * f \tag{2.80}$$

is linear and continuous on $\mathscr{S}(\mathbb{R}^d)$. It thus extends to $\mathscr{S}'(\mathbb{R}^d)$. For suitable h, the formula

$$(g*)'\varphi_h(f) = \varphi_h(g * f) = \int \overline{h}(x) \int g(x-y) f(y) \mathrm{d}y \mathrm{d}x = \varphi_{h * Cg}(f) \tag{2.81}$$

holds with $Cg(x) := \overline{g}(-x)$. We thus define the convolution of g with $\varphi \in \mathscr{S}'(\mathbb{R}^d)$ as

$$g *_{\mathscr{S}'} \varphi := (Cg*)'\varphi(f). \tag{2.82}$$

Definition 2.38. Let $\alpha \in \mathbb{N}^d$. The α-th *distributional derivative* on $\mathscr{S}'(\mathbb{R}^d)$ is defined as $(\partial^\alpha)_{\mathscr{S}'} := (-1)^{|\alpha|}(\partial^\alpha)'$.

Remark 2.39. The definition of $(\partial^\alpha)_{\mathscr{S}'}$ ensures that its action is compatible with the usual derivative and integration by parts: For $g \in \mathscr{S}(\mathbb{R}^d)$

$$\left((\partial^\alpha)_{\mathscr{S}'}\varphi_g\right)(f) = \int \overline{g(x)}(-1)^{|\alpha|}\partial_x^\alpha f(x)\mathrm{d}x = \int \left(\partial_x^\alpha \overline{g}\right)(x) f(x)\mathrm{d}x = \varphi_{\partial^\alpha g}(f). \tag{2.83}$$

For this reason we will not distinguish $(\partial^\alpha)_{\mathscr{S}'}$ from the usual derivative by the notation. The distributional derivative is a local operation: Let $\varphi \in \mathscr{S}'$ have support in the open set $\Omega \subset \mathbb{R}^d$ (i.e.: $\operatorname{supp} f \subset \Omega^c \implies \varphi(f) = 0$), then $\operatorname{supp} \partial^\alpha \varphi \subset \Omega$.

Also note that

$$(\mathscr{F}\partial^\alpha\varphi)(f) = \varphi\left((-1)^{|\alpha|}\partial^\alpha \mathscr{F}^{-1} f\right) = \varphi\left(\mathscr{F}^{-1}(-\mathrm{i})^{|\alpha|}p^\alpha f\right) = \left((-\mathrm{i})^{|\alpha}|p^\alpha \mathscr{F}\varphi\right)(f), \tag{2.84}$$

where multiplication by p^α is defined as M'_{p^α} and in the last step, we used linearity of $f \mapsto \varphi(f)$. Since $g \mapsto \varphi_g$ is anti-linear, this means

$$\varphi_{(\mathrm{i}p)^\alpha \hat{g}} = (-\mathrm{i}p)^\alpha \varphi_{\hat{g}} = \widehat{\partial^\alpha \varphi_g} = \varphi_{\widehat{\partial^\alpha g}}, \tag{2.85}$$

which is consistent with the formula from Proposition 2.9.

2.5. Elliptic PDEs and Sobolev spaces

We can now solve equations such as

$$(\Delta + z)u = f$$

even with $f \in \mathscr{S}'$ by the Fourier transform method (cf. Example 2.11). However, at first we only know that the solution u is an element of $\mathscr{S}'$. We do not, for instance, have a criterion that tells us if $u \in C^k$ and we have found a classical solution.

It is thus important to investigate further these (distributional) solutions. For a special class of constant coefficient linear PDEs, called *elliptic* this can be done quite easily and the regularity of solutions is described precisely by the *Sobolev spaces*.

Definition 2.40. Let

$$P = \sum_{|\alpha|\leq k} a_\alpha \partial^\alpha \tag{2.86}$$

be a constant-coefficient differential operator of order k. The *symbol* of P is the function

$$\sigma_P(p) := \sum_{|\alpha|\leq k} a_\alpha (ip)^\alpha.$$

Since

$$\mathscr{F} Pu = \sigma_P \mathscr{F} u, \tag{2.87}$$

we can solve PDEs as in Example 2.11 if σ_P is invertible for every k. However, the regularity can still be difficult to analyse. The following condition simplifies this enormously:

Definition 2.41. A constant-coefficient differential operator of order k is called *uniformly elliptic* if there exists $c > 0$ so that for all $p \in \mathbb{R}^d$

$$\sum_{|\alpha|=k} a_\alpha (ip)^\alpha \geq c|p|^k. \tag{2.88}$$

We note that this can only hold if $k = 2m$ is even, and only concerns the terms of the highest order in P. The terminology comes from the second order case, where the the condition means that the level sets of σ_P are ellipses.

We will now focus on the simplest elliptic operator $P = -\Delta$, $\sigma_P(p) = -(ip)^2 = p^2$. With some care, results for the general case can be obtained by the same arguments. Our goal is to show that if u is a solution to

$$-\Delta u = f \tag{2.89}$$

and $f \in C^m$, then $u \in C^n$ for an appropriate n (which will depend on the dimension).

Since our method relies on the Fourier transform and this is naturally defined in $\mathscr{S}$, $\mathscr{S}'$ and not C^m, we first need to study subspaces of $\mathscr{S}'$ that classify the regularity of distributions.

Definition 2.42. Let $s \in \mathbb{R}$. The *Sobolev space* of order s is the space

$$H^s(\mathbb{R}^d) := \left\{\varphi \in \mathscr{S}'(\mathbb{R}^d) : \hat{\varphi} \text{ is regular, and } (1+|\cdot|^2)^{s/2}\hat{\varphi} \in L^2(\mathbb{R}^d)\right\} \tag{2.90}$$

with the norm

$$\|\varphi\|_{H^s} = \left\|(1+|\cdot|^2)^{s/2}\hat{\varphi}\right\|_{L^2}. \tag{2.91}$$

Proposition 2.43.

a) We have $H^s(\mathbb{R}^d) \subset H^t(\mathbb{R}^d)$ for $s \geq t$, and in particular $H^s(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ for all $s \geq 0$.

b) If $s \in \mathbb{N}$ is a non-negative integer, then $f \in H^s(\mathbb{R}^d)$ if and only if $f \in L^2(\mathbb{R}^d)$ and $\partial^\alpha f \in L^2(\mathbb{R}^d)$ for all $|\alpha| \leq s$.

Proof. a): Let $s \geq t$. Then

$$\frac{(1+p^2)^t}{(1+p^2)^s} \leq C \tag{2.92}$$

for some $C > 0$. Thus for $f \in H^s$ we have $(1+p^2)^{t/2}\hat{f} \in L^2$, because

$$\begin{aligned}\int (1+p^2)^t|\hat{f}(p)|^2 \mathrm{d}p &= \int \frac{(1+p^2)^t}{(1+p^2)^s}(1+p^2)^s|\hat{f}(p)|^2 \mathrm{d}p \\ &\leq C\int (1+p^2)^s|\hat{f}(p)|^2 \mathrm{d}p = C\|f\|_{H^s}^2. \end{aligned} \tag{2.93}$$

Hence $f \in H^t$ and thus $H^s \subset H^t$. As $H^0 = L^2$ by definition this proves a).

b): Let first $f \in H^m(\mathbb{R}^d)$, $m \in \mathbb{N}$. Then $f \in L^2$ by a) and we have for the derivative in $\mathscr{S}'$

$$\partial^\alpha f = \mathscr{F}^{-1}(ip)^\alpha \hat{f}. \tag{2.94}$$

By Plancherel's Theorem it is thus enough to show that $(ip)^\alpha \hat{f} \in L^2$ for $|\alpha| \leq m$. This now follows from the inequalities

$$\left|(i)^{|\alpha|}p_1^{\alpha_1}\cdots p_d^{\alpha_d}\hat{f}(p)\right|^2 \leq |p|^{2|\alpha|}|f(p)|^2 \leq (1+p^2)^{|\alpha|}|f(p)|^2 \leq (1+p^2)^m|f(p)|^2. \tag{2.95}$$

For the reverse implication, we have by Plancherel that $(ip)^\alpha \hat{f}$ for all $|\alpha| \leq m$ and thus $p^{2\alpha}\hat{|}f(p)|^2 \in L^1$. Now

$$p^{2m} = (p_1^2 + \cdots + p_d^2)^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!}p^{2\alpha} \tag{2.96}$$

by the multinomial theorem, so $p^{2m}|\hat{f}(p)|^2 \in L^1$. This implies that $(1+p^2)^{m/2}\hat{f} \in L^2$ because $(1+p^{2m})/(1+p^2)^m$ is bounded, by the argument of (2.93)

□

Theorem 2.44. *For any $z \in \mathbb{C} \setminus [0, \infty)$ and $\varphi \in \mathscr{S}'(\mathbb{R}^d)$ there exists a unique solution $u \in \mathscr{S}'(\mathbb{R}^d)$ to the equation*

$$(\Delta + z)u = \varphi.$$

Moreover, if $\varphi \in H^s(\mathbb{R}^d)$ for some $s \in \mathbb{R}$ then $u \in H^{s+2}(\mathbb{R}^d)$.

Proof. **Existence**: Since z is not a non-negative real number, $z - p^2 \neq 0$, and $(z-p^2)^{-1}$ is smooth, with bounded derivatives. Hence for $f \in \mathscr{S}(\mathbb{R}^d)$, we have $(z-p^2)^{-1} f \in \mathscr{S}(\mathbb{R}^d)$ and

$$\hat{u}(f) := \hat{\varphi}((z-p^2)^{-1} f) \tag{2.97}$$

defines an element of $\mathscr{S}'(\mathbb{R}^d)$. Setting $u = \mathscr{F}^{-1}\hat{u}$, we have for every $f \in \mathscr{S}$

$$[(\Delta + z)u](f) = u((\Delta + z)f) = \hat{u}(\mathscr{F}(\Delta + z)f) = \hat{u}((z-p^2)\hat{f}) \stackrel{(2.97)}{=} \hat{\varphi}(\hat{f}) = \varphi(f). \tag{2.98}$$

This means that $(\Delta + z)u = \varphi$.

Uniqueness: Let $u, v \in \mathscr{S}'$ be two, possibly different, solutions. Then for all $f \in \mathscr{S}$

$$\hat{u}((z-p^2)f) - \hat{v}((z-p^2)f) = \hat{\varphi}(\hat{f}) - \hat{\varphi}(\hat{f}) = 0. \tag{2.99}$$

Since $f \mapsto (z-p^2)\hat{f}$ is a bijection on $\mathscr{S}(\mathbb{R}^d)$ this implies that $\hat{u} = \hat{v}$, and since the Fourier transform is injective $u = v$.

Regularity: Assume that $\varphi \in H^s(\mathbb{R}^d)$, i.e., $(1+p^2)^{s/2}\hat{\varphi} \in L^2(\mathbb{R}^d)$. First, note that $\hat{\varphi}$ is represented by a measureable function g, i.e.,

$$\hat{\varphi}(f) = \int \overline{g}(p) f(p) \mathrm{d}p. \tag{2.100}$$

Thus $\hat{u}$ is represented by the function $p \mapsto (\overline{z} - p^2)^{-1} g(p)$ and $u \in H^s(\mathbb{R}^d)$, since

$$(1+p^2)^{s/2}|\hat{u}(p)| = (1+p^2)^{s/2}\Big|\frac{g(p)}{\overline{z} - p^2}\Big| \leq C(1+p^2)^{s/2}|g(p)| \in L^2(\mathbb{R}^d). \tag{2.101}$$

Then

$$\begin{aligned}(1+p^2)^{s/2+1}\hat{u} &= (1+p^2)^{s/2}(1+p^2)\hat{u} \\ &= (1+p^2)^{s/2}(1+z)\hat{u} - (1+p^2)^{s/2}\hat{\varphi} \in L^2(\mathbb{R}^d),\end{aligned} \tag{2.102}$$

so $u \in H^{s+2}$. This proves the claim. □

Remark 2.45. We have shown that the linear map $u \mapsto (\Delta + z)u$ from $H^{s+2}(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ is invertible – the inverse in the point φ is given by taking the solution to the equation above.

The link to spaces of differentiable functions is given by the following theorem.

Theorem 2.46 (Sobolev's Lemma)**.** *Let $f \in H^s(\mathbb{R}^d)$ with $s > d/2$. Then f is continuous and for all $m < s - d/2$ we have $f \in C^m(\mathbb{R}^d)$. Moreover, for $s > d/2$ and $|\alpha| \leq m < s - d/2$ there exists a constant so that for all $f \in H^s(\mathbb{R}^d)$*

$$\|\partial_x^\alpha f\|_\infty \leq C\|f\|_{H^s}.$$

Proof. We first show that $f \in H^s(\mathbb{R}^d)$, $s > d/2$ is continuous. By Lemma 2.2, it is sufficient to show that $\hat{f} \in L^1$. This follows from the Cauchy-Schwarz inequality by

$$\int |\hat{f}(p)|\mathrm{d}p = \int (1+p^2)^{-s/2}(1+p^2)^{s/2}|\hat{f}(p)|\mathrm{d}p \leq \|f\|_{H^s}\Big(\int (1+p^2)^{-s}\mathrm{d}p\Big)^{1/2}, \quad (2.103)$$

where the final integral is finite because $2s > d$. Now let $m < s - d/2$ and $|\alpha| \leq m$. Then $(ip)^\alpha \hat{f} \in L^1$, since

$$\int |(ip)^\alpha \hat{f}(p)|\mathrm{d}p \leq \int (1+p^2)^{m/2}|\hat{f}(p)| \leq \|f\|_{H^s}\Big(\int (1+p^2)^{-s+m}\mathrm{d}p\Big)^{1/2}. \quad (2.104)$$

Hence the distributional derivative $\partial^\alpha f \in \mathscr{S}'$ is a continuous function. It remains to show that this equals the usual derivative. We show this for a derivative of order one, the general case follows by repetition of the same argument. Let $\ell \in \{1, \dots, d\}$ and let $g := \mathscr{F}^{-1} i p_\ell \hat{f}$ denote the distributional derivative in direction x_ℓ. Then by the Fourier inversion formula

$$\frac{f(x+\varepsilon e_\ell) - f(x) - \varepsilon g(x)}{\varepsilon} = \frac{1}{(2\pi)^{d/2}} \int \frac{\mathrm{e}^{\mathrm{i}xp + \mathrm{i}\varepsilon p_\ell} - \mathrm{e}^{\mathrm{i}xp} - \mathrm{i}\varepsilon p_\ell \mathrm{e}^{\mathrm{i}px}}{\varepsilon} \hat{f}(p)\mathrm{d}p. \quad (2.105)$$

This converges to zero as $\varepsilon \to 0$ by the dominated convergence theorem, since by the mean-value theorem

$$\Big|\frac{\mathrm{e}^{\mathrm{i}xp + \mathrm{i}\varepsilon p_\ell} - \mathrm{e}^{\mathrm{i}xp} - \mathrm{i}\varepsilon p_\ell \mathrm{e}^{\mathrm{i}px}}{\varepsilon} \hat{f}(p)\Big| \leq 2|p_\ell||\hat{f}(p)|, \quad (2.106)$$

where the right hand side is in $L^1(\mathbb{R}^d)$ and independent of ε. This proves that $g = \partial_{x_\ell} f$, which gives the claim.

□

We can now prove our first regularity result that applies, in particular, to the solutions obtained in Theorem 2.44.

Corollary 2.47. *Let $s \in \mathbb{R}$ and $u \in H^s(\mathbb{R}^d)$. If $-\Delta u = f \in H^t(\mathbb{R}^d)$ for some $t \geq s - 2$, then $u \in H^{t+2}(\mathbb{R}^d)$. If $m < t + 2 - d/2$ is a non-negative integer then also $u \in C^m(\mathbb{R}^d)$.*

Proof. Let $t_1 = \min\{s, t\}$. Since $u \in H^s$, $-\Delta u \in H^{t_1}$, we have

$$\underbrace{(1+p^2)^{t_1/2}\hat{u}(p)}_{\in L^2 \text{ since } t_1 \leq s} + \underbrace{p^2(1+p^2)^{t_1/2}\hat{u}(p)}_{\in L^2 \text{ since } t_1 \leq t} = (1+p^2)^{t_1/2+1}\hat{u}(p) \in L^2(\mathbb{R}^d), \quad (2.107)$$

so $u \in H^{t_1+2}(\mathbb{R}^d)$. If $t_1 = t$ (i.e., $t \leq s$) this proves the claim. Otherwise, we apply the same argument with $s' = t_1 + 2$ and conclude that $u \in H^{t_2+2}(\mathbb{R}^d)$ with $t_2 = \min\{t, s+2\}$. We repeat this until $t_n = \min\{t, s + 2(n-1)\} = t$, and this proves the claim.

The second part $u \in C^m$ follows from Sobolev's Lemma. □

3. Equations with variable coefficients

In the previous chapter we were already able to solve some PDEs, but we were restricted to PDEs with constant coefficients. This restriction allowed us solve the equation using the Fourier transform, which essentially reduces the problem to the calculation of explicit integrals. Of course, in general one cannot hope to solve all PDEs in such an explicit way. For example, the stationary Schrödinger equation

$$-\Delta u(x) + V(x)u(x) = \lambda u(x) \tag{3.1}$$

and the time-dependent Schrödinger equation

$$i\partial_t u(t,x) = -\Delta u(x) + V(x)u(x) \tag{3.2}$$

do not have constant coefficients if the potential $V(x)$ is not constant.

3.1. Linear equations on Hilbert spaces

In this section we will develop general methods for treating linear equations of the form

$$Au = f \tag{3.3}$$

where $u \in X$ for an appropriate vector space X (e.g., $C^\infty(\mathbb{R}^d)$, $\mathscr{S}(\mathbb{R}^d)$, $H^s(\mathbb{R}^d)$), and A is linear.

In most cases we will only consider Hilbert spaces, which additionally have a scalar product.

Definition 3.1. A (complex) Hilbert space $\mathcal{H}$ is a vector space with a scalar product (a positive definite sesquilinear form), such that $\mathcal{H}$ with the norm $\|f\| := \sqrt{\langle f, f\rangle}$ is complete (i.e. a Banach space).

Example 3.2. For $s \in \mathbb{R}$ the Sobolev space $H^s(\mathbb{R}^d)$ is a Hilbert space with the scalar product

$$\langle f, g\rangle_{H^s} = \int (1+p^2)^s \overline{\hat{f}(p)}\hat{g}(p)\mathrm{d}p.$$

Definition 3.3. An orthonormal system (ONS) in $\mathcal{H}$ is a family $\{e_i, i \in I\} \subset \mathcal{H}$, such that

$$\langle e_i, e_j\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

An orthonormal system is called complete (or an orthonormal Hilbert basis) if for every $f \in \mathcal{H}$

$$f = \sum_{i\in I} \langle e_i, f\rangle e_i. \tag{3.4}$$

A Hilbert space is called separable if there exists a countable complete ONS in $\mathcal{H}$. In the following we will only consider separable Hilbert spaces.

Example 3.4. The Sobolev spaces $H^s(\mathbb{R}^d)$ are separable. A complete ONS can be given as follows. First, choose a complete ONS in $L^2(\mathbb{R}^d)$, for example the Hermite functions (here in $d = 1$)

$$e_n(x) := c_n H_n(x) e^{-\frac{1}{2}x^2}, \tag{3.5}$$

where $H_n(x)$ are the Hermite polynomials, c_n normalizing constants, and $n \in \mathbb{N}_0$. A complete ONS of $H^s(\mathbb{R}^d)$ is then given by $e_{n,s}(x) := \mathscr{F}^{-1}(1+p^2)^{-s/2}\hat{e}_n(p)$.

Remark 3.5. Note that separability does not mean that the vector-space dimension of $\mathcal{H}$ is countable (that would require the linear combination to be finite). In fact, the vector space dimension of a Hilbert space is either finite or uncountable by Baire's theorem.

Compactness is very important for solving equations by approximation or fixed-point methods. However, for infinite-dimensional Hilbert spaces we face the following problem:

Proposition 3.6. *Let $B := \overline{B(f,r)}$ be a closed ball in a Hilbert space $\mathcal{H}$. Then B is compact if and only if $\mathcal{H}$ is finite-dimensional.*

Proof. Since translation by f and scaling by r^{-1} is a homeomorphism, it is sufficient to prove the statement for $B := \overline{B(0,1)}$.

If $\mathcal{H}$ has dimension $d < \infty$, then the unit ball is compact because (after choosing an orthonormal basis) it is a closed and bounded subset of $\mathbb{C}^d$.

Assume now that the dimension of $\mathcal{H}$ is infinite and let $f_1 \in \mathcal{H}$ be any vector with $\|f_1\| = 1$. Then $F_1 := \operatorname{span}\{f_1\}$ is a one-dimensional closed subspace of $\mathcal{H}$, and $\mathcal{H} = F \oplus F^\perp$ [FA, Thm.2.3.2]. Since $\dim \mathcal{H} = \infty$, $\dim(F^\perp) = \dim \mathcal{H} - 1 = \infty$, and we can choose $f_2 \in F_1^\perp$ with $\|f_2\| = 1$. Continuing in this way, we find a sequence of vectors f_n, $n \in \mathbb{N}$ satisfying $\|f_n\| = 1$ and $\langle f_n, f_m\rangle = 0$ for $n \neq m$. We thus have for all $n, m \in \mathbb{N}$

$$\|f_n - f_m\|^2 = \|f_n\|^2 - \|f_m\|^2 + 2\mathrm{Re}\langle f_n, f_m\rangle = 2, \tag{3.6}$$

so this sequence cannot contain a convergent subsequence. $\square$

Instead of the standard topology, we will thus sometimes need to consider other, more convenient topologies. In particular, we will use ther weak topology, which is particularly simple on Hilbert spaces. Recall from Functional Analysis [FA, Thm.2.4.1].

Theorem 3.7 (Riesz Representation Theorem)**.** *Let $\mathcal{H}$ be a complex Hilbert space and $\varphi \in \mathcal{H}'$ a continuous linear functional on $\mathcal{H}$. There exists a unique $f \in \mathcal{H}$ so that for all $g \in \mathcal{H}$*

$$\varphi(g) = \langle f, g\rangle.$$

The map

$$\Phi : \mathcal{H} \to \mathcal{H}', \qquad f \mapsto \langle f, \cdot\rangle$$

is an anti-linear isometric isomorphism.

Consequently, weak convergence in a Hilbert space has the following simple characterisation.

Corollary 3.8. *Let $\mathcal{H}$ be a Hilbert space and $f_n \in \mathcal{H}$, $n \in \mathbb{N}$ be a sequence. Then f_n converges weakly to $f \in \mathcal{H}$ if and only*

$$\forall g \in \mathcal{H} : \lim_{n\to\infty} \langle f_n, g\rangle = \langle f, g\rangle.$$

Proof. By defintion, weak convergence means that $\varphi(f_n) \to f$ for all $\varphi \in \mathcal{H}'$, but by Riesz' Theorem $\varphi(f_n) = \langle g, f_n\rangle$ for a unique $g \in \mathcal{H}$. □

Riesz' Theorem also has important consequences for tempered distributions, which are defined as linear functionals. For example, we can show that $(H^s)'$ is naturally identified with H^{-s}.

Corollary 3.9. *Let $\varphi \in \mathscr{S}'(\mathbb{R}^d)$, $s \in \mathbb{R}$ and assume that there exists a constant $C \geq 0$ so that for all $f \in \mathscr{S}(\mathbb{R}^d)$*

$$|\varphi(f)| \leq C\|f\|_{H^s}.$$

Then there exists $h \in H^{-s}(\mathbb{R}^d)$ so that

$$\varphi(f) = \int \overline{\hat{h}(p)}\hat{f}(p)\mathrm{d}p.$$

Proof. By the assumed inequality, the linear map $\varphi : \mathscr{S} \to \mathbb{C}$ is bounded with respect to the H^s-norm. Since $\mathscr{S}$ is dense in H^s (Problem 29), we can thus extend φ uniquely to a continuous linear functional on $H^s(\mathbb{R}^d)$ by the B.L.T Theorem. By the Riesz Representation Theorem, there exists $g \in H^s(\mathbb{R}^d)$ so that

$$\varphi(f) = \int (1+p^2)^s\overline{g(p)}\hat{f}(p)\mathrm{d}p = \int \overline{\hat{h}(p)}\hat{f}(p)\mathrm{d}p \tag{3.7}$$

with $\hat{h}(p) := (1+p^2)^s\hat{g}(p)$, which is clearly an element of $H^{-s}(\mathbb{R}^d)$. □

Example 3.10. Let e_n, $n \in \mathbb{N}$ be an ONS in $\mathcal{H}$. Then e_n converges to zero weakly as $n \to \infty$. Indeed, for any $g \in \mathcal{H}$ we have the Bessel inequality

$$\sum_{n=1}^{\infty} |\langle g, e_n\rangle|^2 \leq \|g\|^2, \tag{3.8}$$

so $\langle g, e_n\rangle$ converges to zero.

As seen above, there are bounded sequences in $H^s(\mathbb{R}^d)$ that do not have convergent subsequences. However, with the notion of weak convergence we can still find some sort of limit.

Theorem 3.11. *Let $\mathcal{H}$ be a separable Hilbert space. Any bounded sequence $f_n \in \mathcal{H}$, $n \in \mathbb{N}$, has a weakly convergent subsequence.*

Proof. Let e_n, $n \in \mathbb{N}$, be a complete ONS in $\mathcal{H}$. For every n, the sequence $k \mapsto \langle e_n, f_k \rangle$ is a bounded sequence in $\mathbb{C}$ (and thus has a convergent subsequence), since

$$|\langle e_n, f_k \rangle| \leq \|f_k\| \leq \sup_k \|f_k\| < \infty. \tag{3.9}$$

We will extract a joint convergent subsequence by a diagonal argument. Start with $n = 1$ by extracting a convergent subsequence, i.e., an infinite subset $S_1 \subset \mathbb{N}$ with

$$\lim_{\substack{k \to \infty \\ k \in S_1}} \langle e_1, f_k \rangle = c_1 \in \mathbb{C}. \tag{3.10}$$

The sequence $\langle e_2, f_k \rangle$, $k \in S_1$, is obviously bounded, so we can again extract a convergent subsequence $S_2 \subset S_1 \subset \mathbb{N}$. By repeating this argument, we obtain infinite sets S_j, $j \in \mathbb{N}$ with $S_j \subset S_\ell$ if $j > \ell$.

Now let k_j be the j-th element of S_j (i.e., k_1 is the smallest element of S_1, k_2 the second of S_2, etc.). Then we have

$$\lim_{j \to \infty} k_j = \infty \tag{3.11}$$

$$k_j \in \bigcap_{\ell \leq j} S_\ell. \tag{3.12}$$

Consequently for all $n \in \mathbb{N}$

$$\lim_{j \to \infty} \langle e_n, f_{k_j} \rangle = c_n, \tag{3.13}$$

because $k_j \in S_n$ for $j \geq n$.

We now claim that f_{k_j} converges weakly to

$$f := \sum_{n=1}^{\infty} c_n e_n. \tag{3.14}$$

First note that, by Fatou's Lemma and Parseval's identity

$$\sum_{n=1}^{\infty} |c_n|^2 \leq \operatorname{liminf}_{j \to \infty} \sum_{n=1}^{\infty} |\langle e_n, f_{k_j} \rangle|^2 = \operatorname{liminf}_{j \to \infty} \|f_{k_j}\|^2 \leq \sup_k \|f_k\|^2 < \infty. \tag{3.15}$$

Hence $(c_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$ and $f \in \mathcal{H}$ by Parseval's identity. Now for any $g \in \mathcal{H}$ and $N \in N$

$$\begin{aligned} |\langle g, f_{k_j} - f \rangle| &\leq \sum_{n=1}^{\infty} |\langle g, e_n \rangle| |\langle e_n, f_{k_j} - f \rangle| \\ &\leq \sum_{n=1}^{N} |\langle g, e_n \rangle| |\langle e_n, f_{k_j} - f \rangle| + \left(\sum_{n=N+1}^{\infty} |\langle g, e_n \rangle|^2 \right)^{1/2} \|f_{k_j} - f\| \\ &\leq \sum_{n=1}^{N} |\langle g, e_n \rangle| |\langle e_n, f_{k_j} - f \rangle| + \left(\sum_{n=N+1}^{\infty} |\langle g, e_n \rangle|^2 \right)^{1/2} 2 \sup_k \|f_k\|. \end{aligned} \tag{3.16}$$

Let $\varepsilon > 0$. Then, since $\langle g, e_n \rangle \in \ell^2$ and thus

$$\lim_{N\to\infty} \sum_{n=N+1}^{\infty} |\langle g, e_n \rangle|^2 = \lim_{N\to\infty} \left(\|g\|^2 - \sum_{n=1}^{N} |\langle g, e_n \rangle|^2 \right) = 0, \tag{3.17}$$

we can choose $N(\varepsilon)$ so that the second term is less than $\varepsilon/2$. The first term is then a finite sum of sequences that all converge to zero as $j \to \infty$, so we can make it smaller than $\varepsilon/2$ by choosing $j \geq j_0(N, \varepsilon)$ large enough. This proves the claim. □

We will now turn our focus to the operators $\mathrm{B}(\mathcal{H})$ on a separable complex Hilbert space $\mathcal{H}$.

Recall from [FA, Prop.3.1.9] that the adjoint A^* of $A \in \mathrm{B}(\mathcal{H})$ is the unique operator defined by the relation

$$\forall f, g \in \mathcal{H} : \langle f, Ag \rangle = \langle A^* f, g \rangle. \tag{3.18}$$

The following definition generalises well-known notions for matrices to $\mathrm{B}(\mathcal{H})$.

Definition 3.12. Let $A \in \mathrm{B}(\mathcal{H})$.

a) A is called *self-adjoint* if $A^* = A$;

b) A is called *unitary* if $A^*A = 1 = AA^*$;

c) A is called *normal* if $A^*A = AA^*$.

Question 3.13. Which of the following operators are normal and/or self-adjoint, unitary?

a) $M_g f = gf$ with $g \in L^\infty(\mathbb{R}^d)$ on $L^2(\mathbb{R}^d)$;

b) $T_v f = f(\cdot + v)$ with $v \in \mathbb{R}^d$ on $L^2(\mathbb{R}^d)$;

c) $T_t f = f(\cdot + t)$ with $t > 0$ on $L^2(\mathbb{R}_+)$.

Example 3.14. Let $f \in L^2(\mathbb{R}^d)$ and $u(t, x)$ be the solution of the heat equation (cf. Problem 16)

$$\begin{cases} \partial_t u(t, x) = \Delta u(t, x), & (t, x) \in (0, \infty) \times \mathbb{R}^d \\ u(0, x) = f(x). \end{cases}$$

Then $T_t f := u(t, \cdot)$, $t \geq 0$, is self-adjoint (and hence normal) on $L^2(\mathbb{R}^d)$, but not unitary for $t > 0$. To see this, write for $t > 0$

$$u(t, x) = \int_{\mathbb{R}^d} E_t(x - y) f(y) \mathrm{d}y = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/(4t)} f(y) \mathrm{d}y. \tag{3.19}$$

Then, because $E(t, x - y)$ is real and symmetric under exchange of x, y, we have

$$\langle g, E_t * f \rangle = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \overline{g(x)} \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/(4t)} f(y) \mathrm{d}y \mathrm{d}x = \langle E_t * g, f \rangle, \tag{3.20}$$

so $T_t^* = T_t$. We have

$$T_t^* T_t = T_t^2 = T_{2t} \tag{3.21}$$

because $u(t+s)$ solves the heat equation with $u(t+0) = u(t)$. Now $T_{2t} \neq 1$ for $t > 0$ so T_t is not unitary. We also have $\|T_t f\| \leq \|f\|$ hence we say that T_t is a *contraction* (in fact, the inequality is strict because $\lambda = 1$ is not an eigenvalue of T_t, but the bound cannot be improved uniformly in f).

3.2. An elliptic equation with variable coefficients

As an application of our results we can now study the elliptic equation

$$-\operatorname{div} M(x)\nabla u(x) + \lambda u(x) = f(x) \tag{3.22}$$

for a non-trivial coefficients matrix M. We assume that M is uniformly elliptic, that is, there exists $a > 0$ so that for all $x \in \mathbb{R}^d$, $v \in \mathbb{C}^d$

$$\langle v, M(x)v\rangle_{\mathbb{C}^d} \geq a\|v\|^2. \tag{3.23}$$

Instead of studying the equation (3.22) directly, we will first consider its weak form. Assume that $f \in L^2(\mathbb{R}^d)$, $\lambda \in \mathbb{R}$, and $u \in L^2(\mathbb{R}^d)$ solves (3.22). Then for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$ we have

$$\begin{aligned} \langle f, \varphi\rangle_{L^2(\mathbb{R}^d)} &= \langle -\operatorname{div} M(x)\nabla u + \lambda u, \varphi\rangle_{L^2} \\ &= \int \langle M(x)\nabla u(x), \nabla\varphi(x)\rangle_{\mathbb{C}^d} \mathrm{d}x + \lambda\langle u, \varphi\rangle_{L^2} \\ &= \langle M(x)\nabla u(x), \nabla\varphi(x)\rangle_{L^2(\mathbb{R}^d,\mathbb{C}^d)} + \lambda\langle u, \varphi\rangle_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{3.24}$$

If M is bounded, the latter expression is well defined for $u, \varphi \in H^1(\mathbb{R}^d)$. We thus call $u \in H^1(\mathbb{R}^d)$ a weak solution to (3.22) if

$$\forall\varphi \in H^1(\mathbb{R}^d) : \langle M\nabla u, \nabla\varphi\rangle_{L^2(\mathbb{R}^d,\mathbb{C}^d)} + \lambda\langle u, \varphi\rangle_{L^2(\mathbb{R}^d)} = \langle f, \varphi\rangle_{L^2(\mathbb{R}^d)}. \tag{3.25}$$

Theorem 3.15. *Let $M \in L^\infty(\mathbb{R}^d, \mathrm{B}(\mathbb{C}^d))$ be uniformly elliptic. Then for every $\lambda > 0$ and $f \in L^2(\mathbb{R}^d)$ there exists a unique solution $u \in H^1(\mathbb{R}^d)$ to* (3.25).

Proof. For $f, g \in H^1(\mathbb{R}^d)$ denote

$$\ll f, g \gg := \lambda\langle f, g\rangle_{L^2(\mathbb{R}^d)} + \langle M\nabla f, \nabla g\rangle_{L^2(\mathbb{R}^d,\mathbb{C}^d)}. \tag{3.26}$$

This is a scalar product on $H^1(\mathbb{R}^d)$. By ellipticity of M, we have

$$\ll f, f \gg = \lambda\|f\|_{L^2}^2 + \int \underbrace{\langle M(x)\nabla f(x), \nabla f(x)\rangle}_{\geq a\|\nabla f(x)\|^2} \geq \lambda\|f\|_{L^2}^2 + a\|\nabla f\|_{L^2}^2 \geq \min\{a, \lambda\}\|f\|_{H^1}^2. \tag{3.27}$$

On the other hand, by boundedness of M,

$$\ll f, f \gg \leq \lambda \|f\|_{L^2}^2 + \|M\|_{L^\infty} \|\nabla f\|_{L^2}^2 \leq \max\{\lambda, \|M\|_\infty\} \|f\|_{H^1}^2. \tag{3.28}$$

The norm induced by $\ll f, g \gg$ is thus equivalent to the H^1-norm, so H^1 equipped with this scalar product is complete, i.e. a Hilbert space.

The right hand side of the equation satisfies

$$|\langle f, \varphi \rangle| \leq \|f\|_{L^2} \|\varphi\|_{L^2} \leq \|f\|_{L^2} \|\varphi\|_{H^1}. \tag{3.29}$$

The map $\varphi \mapsto \langle f, \varphi \rangle$ is thus a continuous linear functional on $H^1(\mathbb{R}^d)$. By the Riesz Representation Theorem there exists a unique $u \in H^1(\mathbb{R}^d)$ so that

$$\langle f, \varphi \rangle = \ll u, \varphi \gg, \tag{3.30}$$

i.e, u is the unique solution to (3.25). □

We want to establish that, when the coefficient matrix M is sufficiently regular, the weak solution obtained in this theorem is an element of $H^2(\mathbb{R}^d)$ and solves the equation (3.22) in the sense of equality in $L^2(\mathbb{R}^d)$.

To this end, we need the following Lemma on the difference quotients.

Lemma 3.16. *Define for $0 \neq h \in \mathbb{R}^d$ an operator $D_h \in \mathrm{B}(L^2(\mathbb{R}^d))$ by*

$$(D_h f)(x) = \frac{f(x+h) - f(x)}{|h|}.$$

a) If $f \in H^1(\mathbb{R}^d)$, then for all $h \in \mathbb{R}^d$: $\|D_h f\| \leq \|\nabla f\|$.

b) If $f \in L^2(\mathbb{R}^d)$ and $\sup_{0 \neq h \in \mathbb{R}^d} \|D_h f\| < \infty$, then $f \in H^1(\mathbb{R}^d)$.

Proof. a) Assume first that $f \in \mathscr{S}(\mathbb{R}^d)$. Then by the fundamental theorem of calculus

$$|D_h f(x)| = \left| \frac{1}{|h|} \int_0^1 h \cdot \nabla f(x + th) \mathrm{d}t \right| \leq \int_0^1 |\nabla f(x + th)| \mathrm{d}t. \tag{3.31}$$

Thus by Cauchy-Schwarz

$$\|D_h f\|^2 \leq \int_0^1 \int_0^1 \int_{\mathbb{R}^d} |\nabla f(x + th)| |\nabla f(x + sh)| \mathrm{d}x \mathrm{d}t \mathrm{d}s \leq \|\nabla f\|^2. \tag{3.32}$$

Since $\mathscr{S}$ is dense in $H^1(\mathbb{R}^d)$, the bounded linear maps $D_h : \mathscr{S} \to L^2$ can be extended to $H^1(\mathbb{R}^d)$ with the same norm, so the inequality still holds for $f \in H^1(\mathbb{R}^d)$. This proves a).

b) Let $i \in \{1, \ldots, d\}$, $n \in \mathbb{N}$ and set $h_n = n^{-1} e_i$. By hypothesis, the sequence $D_{h_n} f$ is bounded in $L^2(\mathbb{R}^d)$. Hence by Theorem 3.11 it has a weakly convergent subsequence, which we denote by the same symbols. Let $g \in L^2(\mathbb{R}^d)$ denote the weak limit and let $\varphi \in \mathscr{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. Then by a change of variables and dominated convergence

$$\begin{aligned}\langle \varphi, g\rangle &= \lim_{n\to\infty} \langle \varphi, D_{h_n} f\rangle \\ &= \lim_{n\to\infty} n \int \overline{\varphi}(x)(f(x+n^{-1}e_i) - f(x))\mathrm{d}x \\ &= \lim_{n\to\infty} \int n(\overline{\varphi}(x-n^{-1}e_n) - \overline{\varphi}(x))f(x)\mathrm{d}x \\ &= \int -(\partial_i \overline{\varphi}(x))f(x)\mathrm{d}x. \end{aligned} \tag{3.33}$$

Hence g coincides with $\partial_i f$ in $\mathscr{S}'(\mathbb{R}^d)$, whence $\partial_i f \in L^2(\mathbb{R}^d)$. In view of Proposition 2.43 this shows that $f \in H^1(\mathbb{R}^d)$. □

Theorem 3.17. *Assume the hypothesis of Theorem 3.15 and additionally that* $M \in C^1(\mathbb{R}^d, \mathrm{B}(\mathbb{C}^d))$ *and that* ∇M *is bounded. Let* u *be the weak solution to* (3.25)*, then* $u \in H^2(\mathbb{R}^d)$ *and* (3.22) *holds in* $L^2(\mathbb{R}^d)$*.*

Proof. The idea is to take the derivative of the equation, but since we do not know a priori that this makes sense, we rather consider difference quotients D_h as above.

We know that $u \in H^1(\mathbb{R}^d)$, so we may take $\varphi = D_{-h}D_h u$ in (3.25). Note that we have the following identities:

$$D_{-h}^* = D_h \tag{3.34}$$

$$\nabla D_h f = D_h \nabla f \tag{3.35}$$

$$D_h(fg) = (\tau_h f)D_h g + g D_h f, \tag{3.36}$$

where $\tau_h f(x) = f(x+h)$. With this, we find from (3.25)

$$\begin{aligned}\langle f, D_{-h}D_h u\rangle &= \langle D_h M\nabla u, D_h \nabla u\rangle + \lambda\langle u, D_{-h}D_h u\rangle \\ &= \langle (\tau_h M)D_h\nabla u, D_h\nabla u\rangle + \langle (D_h M)\nabla u, D_h \nabla u\rangle + \lambda\langle u, D_{-h}D_h u\rangle. \end{aligned}\tag{3.37}$$

Using that M is elliptic, we obtain from this and Lemma 3.16

$$\begin{aligned} a\|D_h\nabla u\|^2 &\leq \langle (\tau_h M)D_h\nabla u, D_h\nabla u\rangle \\ &\overset{(3.37)}{\leq} |\langle f, D_{-h}D_h u\rangle| + |\langle (D_h M)\nabla u, D_h\nabla u\rangle| + \lambda\|u\|\|D_{-h}D_h u\| \\ &\leq (\|f\| + \lambda\|u\|)\|\nabla D_h u\| + \|D_h M\|_{L^\infty}\|\nabla u\|\|D_h\nabla u\|. \end{aligned}\tag{3.38}$$

Now we have

$$\|D_h M\|_{L^\infty} = \left\| \int_0^1 \frac{h}{|h|}\nabla M(x+th)\mathrm{d}t \right\|_{L^\infty} \leq \|\nabla M\|_{L^\infty}, \tag{3.39}$$

so dividing (3.38) by $\|\nabla D_h u\|$ yields

$$a\|D_h\nabla u\| \leq \|f\| + \lambda\|u\| + \|\nabla M\|_{L^\infty}\|\nabla u\|. \tag{3.40}$$

By Lemma 3.16 this proves that $\nabla u \in H^1(\mathbb{R}^d, \mathbb{C}^d)$, so, by Proposition 2.43, $u \in H^2(\mathbb{R}^d)$.

By Exercise 33 we thus have $M\nabla u \in H^1(\mathbb{R}^d, \mathbb{C}^d)$, and obtain from the weak form of the equation (3.25)

$$\langle f, \varphi \rangle = \langle M\nabla u, \nabla \varphi \rangle + \lambda \langle u, \varphi \rangle \tag{3.41}$$

$$= \langle - \operatorname{div} M\nabla u + \lambda u, \varphi \rangle \tag{3.42}$$

for all $\varphi \in H^1(\mathbb{R}^d)$. Since the latter is dense in $L^2(\mathbb{R}^d)$ this implies that

$$f + \operatorname{div} M\nabla u + \lambda u \in (H^1(\mathbb{R}^d))^{\perp} = \{0\}, \tag{3.43}$$

that is, equation (3.22) holds. □

Remark 3.18. If the coefficients M have $k+1$ bounded derivatives and $f \in H^k(\mathbb{R}^d)$ we can iterate the reasoning of Theorem 3.17 and obtain $u \in H^{k+2}(\mathbb{R}^d)$.

4. Linear evolution equations

In this chapter, we will study the "initial value problem", also called the abstract Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = Au \\ u(0) = u_0 \end{cases} \tag{4.1}$$

for suitable densely-defined operator A, $D(A)$ on $\mathcal{H}$.

4.1. The exponential of a bounded operator

The simplest case for (4.1) is when $A \in \mathrm{B}(\mathcal{H})$ is bounded (and thus $D(A) = \mathcal{H}$). This case is very similar to linear ODEs.

Lemma 4.1. *Let $A \in \mathrm{B}(\mathcal{H})$. Then the exponential series*

$$\mathrm{e}^A := \sum_{j=0}^{\infty} \frac{A^j}{j!}$$

converges in $\mathrm{B}(\mathcal{H})$*, and*

$$\|\mathrm{e}^A\| \leq \mathrm{e}^{\|A\|}.$$

Proof. We have

$$\|A^j\| \leq \|A\|^j, \tag{4.2}$$

and thus

$$\Big\| \sum_{j=n}^{m} \frac{A^j}{j!} \Big\| \leq \sum_{j=n}^{m} \frac{\|A\|^j}{j!} \leq \mathrm{e}^{\|A\|} - \sum_{j=0}^{n-1} \frac{\|A\|^j}{j!}. \tag{4.3}$$

The right hand side converges to zero for $n \to \infty$ since the exponential series of real numbers converges. The sequence of partial sums is thus Cauchy in $\mathrm{B}(\mathcal{H})$ and by completeness it has a limit e^A. □

Theorem 4.2. *Let $A \in \mathrm{B}(\mathcal{H})$. For every $u_0 \in \mathcal{H}$,*

$$u = \mathrm{e}^{tA} u_0 \in C^\infty(\mathbb{R}, \mathcal{H}) \tag{4.4}$$

solves the the Cauchy problem (4.1)*. This solution is the unique maximal solution to* (4.1)*, that is, if $v \in C^1((-\varepsilon, \varepsilon), \mathcal{H})$ solves* (4.1)*, then $v = u|_{(-\varepsilon,\varepsilon)}$.*

Proof. This proof is essentially the same as for linear ODEs.

To start with, we have

$$u(0) = e^0 u_0 = u_0. \tag{4.5}$$

We now show that $u \in C^1(\mathbb{R}, \mathcal{H})$ with derivative Au, so u solves (4.1). We have

$$\begin{aligned}
\frac{u(t+h) - u(t) - hAu(t)}{h} &= \frac{1}{h}\sum_{j=0}^{\infty} \frac{(t+h)^j - t^j}{j!} A^j u_0 - A \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j u_0 \\
&= \frac{1}{h}\sum_{j=0}^{\infty} \frac{(t+h)^{j+1} - t^{j+1} - (j+1)ht^j}{(j+1)!} A^{j+1} u_0
\end{aligned} \tag{4.6}$$

The term with $j = 0$ vanishes, and for $j \geq 1$ we have by the mean value theorem

$$\begin{aligned}
(t+h)^{j+1} - t^{j+1} - (j+1)ht^j &\overset{\tau_j \in [t, t+h]}{=} h(j+1)(\tau^j - t^j) \\
&\overset{\sigma_j \in [t, \tau_j]}{=} h(\tau_j - t) j (j+1) \sigma^{j-1},
\end{aligned} \tag{4.7}$$

so since $|\tau_j - t| \leq |h|$, $|\sigma_j| \leq |t| + |h|$, we have

$$\left| \frac{\mathrm{e}^{(t+h)A} u_0 - \mathrm{e}^{tA} u_0 - h}{h} \right| \leq |h| \|A\|^2 \mathrm{e}^{(|t|+|h|)\|A\|} \|u_0\|, \tag{4.8}$$

which converges to zero as $h \to 0$, so

$$\frac{\mathrm{d}\mathrm{e}^{At} u_0}{\mathrm{d}t} = A\mathrm{e}^{At} u_0, \tag{4.9}$$

which proves the claim. Since $u'(t) = \mathrm{e}^{At} A u_0$ has the same form, we can iterate this and obtain that $u \in C^\infty(\mathbb{R}, \mathcal{H})$.

Now assume that $v : (-\varepsilon, \varepsilon) \to \mathcal{H}$ solves (4.1). Then for all $|t| \leq \varepsilon$

$$\begin{aligned}
\|u(t) - v(t)\|^2 &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \|u(s) - v(s)\|^2 \mathrm{d}s \\
&= \int_0^t 2\mathrm{Re}\langle (u(s) - v(s)), A(u(s) - v(s)) \rangle \mathrm{d}s \\
&\leq \int_0^t 2\|A\| \|u(s) - v(s)\|^2 \mathrm{d}s.
\end{aligned} \tag{4.10}$$

Thus by Grønwall's inequality, this is less than the solution to the equation $x' = 2\|A\|x$, $x(0) = 0$, which vanishes. This proves uniqueness of u. ☐

Corollary 4.3. *For $t, s \in \mathbb{R}$ we have* $\mathrm{e}^{(t+s)A} = \mathrm{e}^{tA}\mathrm{e}^{sA}$

Proof. While this can also be seen from the exponential series, it follows immediately from uniqueness of the solutions to (4.1) by the following argument. Let $f \in \mathcal{H}$ and consider the functions

$$\begin{aligned}
u(t) &= \mathrm{e}^{(t+s)A} f \\
v(t) &= \mathrm{e}^{tA}\mathrm{e}^{sA} f.
\end{aligned} \tag{4.11}$$

Both solve (4.1) with initial condition $v(0) = \mathrm{e}^{As} f = u(0)$, so they must be equal. Since f was arbitary this proves equality of the operators. ☐

4.2. The Hille-Yosida theorem

In this section we will prove a theorem that ensures the existence and uniqueness of solutions to the abstract Cauchy problem under suitable hypothesis on the generators A.

The key condition is that the generator should be (maximal) dissipative, which excludes directions of exponential growth for the solutions. With this condition the solutions will satisfy $\|u(t)\| \leq \|u_0\|$, instead of the general bound $\|u(t)\| \leq \mathrm{e}^{\|A\|}$, $A \in \mathrm{B}(\mathcal{H})$ which cannot be generalised to unbounded operators.

Definition 4.4. Let A, $D(A)$ be a densely defined operator on a Hilbert space $\mathcal{H}$, i.e., $A : D(A) \to \mathcal{H}$ is a linear map and $\overline{D(A)} = \mathcal{H}$. The operator A, $D(A)$ is called *dissipative* if

$$\forall f \in D(A) : \mathrm{Re}\langle f, Af\rangle \leq 0. \tag{4.12}$$

The operator is called *maximal dissipative* if additionally $A - 1$ is surjective, i.e.

$$\mathrm{ran}(A - 1) = \mathcal{H}. \tag{4.13}$$

Question 4.5. Which of the following operators with domain $D = H^2(\mathbb{R}^d)$ is (maximal) dissipative?

1) $A_1 = \Delta$;

2) $A_2 = -\Delta$;

3) $A_3 = \mathrm{i}\Delta$.

Example 4.6. The operator $A = \operatorname{div} M\nabla$, $D(A) = H^2(\mathbb{R}^d)$ from Section 3.2 with $M : \mathbb{R}^d \to \mathbb{C}^{d\times d}$ positive definite is dissipative since

$$\langle f, Af\rangle = -\int_{\mathbb{R}^d} \langle \nabla f(x), M(x)\nabla f(x)\rangle \mathrm{d}x \leq 0. \tag{4.14}$$

If M is uniformly elliptic (cf. (3.23)) and satisfies the hypothesis of Theorem 3.17 then it is maximal dissipative, since $\lambda - A$ is onto for all $\lambda > 0$ by Theorem 3.17.

Recall from [FA]:

Definition 4.7. Let $H, D(H)$ be densely defined on $\mathcal{H}$. We define the *adjoint* H^*, $D(H^*)$ by

$$\begin{aligned} D(H^*) &:= \{g \in \mathcal{H} : \exists h_g \in \mathcal{H}\, \forall f \in D(H) : \langle Hf, g\rangle = \langle f, h_g\rangle\}, \\ H^* &: D(H^*) \to \mathcal{H}, \\ H^* g &:= h_g \end{aligned}$$

The operator H is called *symmetric* if H^* extends H, i.e.,

$$\forall f, g \in D(H) : \langle f, Hg\rangle = \langle Hf, g\rangle.$$

The operator H is called *self-adjoint* if $H = H^*$.

Proposition 4.8. *Let $(H, D(H))$ be a symmetric operator on $\mathcal{H}$. Then $A = \mathrm{i}H$ is dissipative. If H is self-adjoint, then $A = \mathrm{i}H$ is maximal dissipative.*

Proof. As H is symmetric,

$$\langle f, Af \rangle = \mathrm{i}\langle f, Hf \rangle \in \mathrm{i}\mathbb{R}, \tag{4.15}$$

so A is dissipative. Now assume that H is self-adjoint. Then by [FA, Prop.4.3.4]

$$\mathrm{ran}(A - 1) = \mathrm{ran}(\mathrm{i}H - 1) = \mathrm{ran}(H + \mathrm{i}) = \mathcal{H}, \tag{4.16}$$

so A is maximal dissipative. □

Recall from [FA]:

Definition 4.9. The operator $A, D(A)$ is called *closed* if its graph

$$\mathscr{G}(A) := \{(f, Af) : f \in D(A)\} \subset D(A) \times \mathcal{H} \subset \mathcal{H} \times \mathcal{H} \tag{4.17}$$

is closed in $\mathcal{H} \times \mathcal{H}$, i.e., for any sequence $(f_n)_{n\in\mathbb{N}}$ in $D(A)$ such that f_n converges to $f \in \mathcal{H}$ and Af_n converges to $g \in \mathcal{H}$, it holds that $f \in D(A)$ and $Af = g$.

The operator A, $D(A)$ is called *closable* if it has a closed extension.

Definition 4.10. Let $A, D(A)$ be densely defined on $\mathcal{H}$ and closed. The set

$$\rho(A) := \{z \in \mathbb{C} : A - z : D(A) \to \mathcal{H} \text{ is bijective, and } (A - z)^{-1} \text{ is bounded}\} \tag{4.18}$$

is called the *resolvent set* of A. For $z \in \rho(A)$ the operator

$$R_z(A) := (A - z)^{-1} \tag{4.19}$$

is called the *resolvent*. The complement $\sigma(A) := \mathbb{C} \setminus \rho(A)$ is the *spectrum* of A. It is composed of

- The *point spectrum*

$$\sigma_p(A) := \{z \in \mathbb{C} : A - z \text{ is not one-to-one}\}$$

- The *continuous spectrum*

$$\sigma_c(A) := \{z \in \mathbb{C} : A - z \text{ is one-to-one, } \mathrm{ran}(A - z) \neq \mathcal{H} \text{ but } \overline{\mathrm{ran}(A - z)} = \mathcal{H}\}$$

- The *residual spectrum*

$$\sigma_r(A) := \{z \in \mathbb{C} : A - z \text{ is one-to-one but } \overline{\mathrm{ran}(A - z)} \neq \mathcal{H}\}.$$

Proposition 4.11. *Let A, $D(A)$ be dissipative.*

a) For every $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, $A - z$ is injective;

b) If there exists $\lambda_0 > 0$ such that $A - \lambda_0$ is onto, then A is closed, the spectrum

$$\sigma(A) \subset \{z \in \mathbb{C} : \mathrm{Re} z \leq 0\},$$

and for all z with $\mathrm{Re} z > 0$

$$\|R_z(A)\| \leq \frac{1}{\mathrm{Re} z}.$$

Proof. a) Let $z = \lambda + \mathrm{i}\mu$ with $\lambda > 0$. Clearly, $A - \mathrm{i}\mu$ is also dissipative, so it is sufficient to prove the statement for $\mu = 0$. We have for $f \in D(A)$, $\lambda > 0$

$$\begin{aligned} \|(A-\lambda)f\|^2 &= \|Af\|^2 + \lambda^2\|f\|^2 - 2\lambda \mathrm{Re}\langle f, Af, \rangle \\ &\geq \lambda^2 \|f\|^2. \end{aligned} \tag{4.20}$$

This shows that $(A - \lambda)f = 0 \implies f = 0$, so $\ker(A - \lambda) = \{0\}$.

b) By a) and the fact that $A - \lambda_0$ is onto, we have that $A - \lambda_0$ is bijective. Applying (4.20) with $\lambda = \lambda_0$ and $f = (A - \lambda)^{-1} g$, $g \in \mathcal{H}$ we find

$$\|g\|^2 \geq \lambda_0^2 \|(A - \lambda_0)^{-1} g\|^2, \tag{4.21}$$

so $(A - \lambda_0)^{-1} \in \mathrm{B}(\mathcal{H})$ and $\lambda_0 \in \rho(A)$. By [FA, Prop.4.1.4], A is closed.

We also have $\|R_{\lambda_0}(A)\| \leq \lambda_0^{-1}$, and by [FA, Thm.4.2.5] we thus have $B(\lambda_0, \lambda_0) \subset \rho(A)$. Indeed, for $|z - \lambda_0| < \lambda_0$,

$$\begin{aligned} R_z(A) &= \Big((1 - R_{\lambda_0}(A)(z - \lambda_0))(A - \lambda_0) \Big)^{-1} = R_{\lambda_0}(A) \Big(1 - R_{\lambda_0}(A)(z - \lambda_0) \Big)^{-1} \\ &= \sum_{k=0}^{\infty} R_{z_0}(A)^{k+1} (z - \lambda_0)^k, \end{aligned} \tag{4.22}$$

where the sum converges in the operator norm.

Let $z = \lambda + \mathrm{i}\mu \in \rho(A)$ with $\lambda > 0$. Applying (4.20) to $A_\mu = A - \mathrm{i}\mu$ with $f = R_z(A)g$, $g \in \mathcal{H}$ we find

$$\|g\|^2 \geq \mathrm{Re}(z)^2 \|R_z(A) g\|^2. \tag{4.23}$$

Using this bound on the norm of the resolvent, we can then expand around additional points and enlarge the known resolvent set until it covers the (open) right half plane. The spectrum is thus contained in the (closed) left half plane (compare [FA, Thm.4.2.9]). □

Corollary 4.12. *Let A, $D(A)$ be dissipative. The following are equivalent*

1) A is maximal dissipative;

2) $A - z$ is surjective for all $z \in \mathbb{C}$ with $\mathrm{Re} z > 0$*;*

3) $A - \lambda$ is surjective for some $\lambda > 0$.

Example 4.13. Let $H = \operatorname{div} M\nabla$, $D(H) = H^2(\mathbb{R}^d)$ from Section 3.2 with $M : \mathbb{R}^d \to \mathbb{C}^{d\times d}$ positive definite, uniformly elliptic. Then $A = \pm \mathrm{i}H$ is maximal dissipative. Indeed, since M is self-adjoint,

$$\int_{\mathbb{R}^d} \langle \nabla f(x), M(x)\nabla f(x)\rangle \mathrm{d}x \tag{4.24}$$

is real, so $\mathrm{Re}\langle f, Af\rangle = 0$. Moreover, since H is maximal dissipative by Example 4.6, Corollary 4.12 implies that $H - 1 \pm \mathrm{i}$ is onto. Then, so is

$$A - 1 \pm \mathrm{i} = \mp \mathrm{i}(H - 1 \mp \mathrm{i}). \tag{4.25}$$

Note that, since H is symmetric and $H - 1 \pm \mathrm{i}$ is onto, the self-adjointness criterion [FA, Prop.4.3.4] shows that $H - 1$, and thus also H, is self-adjoint, so this is a case of Proposition 4.8.

We will now work toward solving the abstract Cauchy problem (4.1) for a generator $A, D(A)$ that is maximal dissipative and an initial contition $u_0 \in D(A)$.

The idea is to use the spectral information on A we have obtained and approximate A by bounded operators, the so-called Yosida-approximants,

$$A_n := -nAR_n(A) \tag{4.26}$$

Lemma 4.14. *Let $A, D(A)$ be maximal dissipative and define A_n by (4.26) for $n \in \mathbb{N}$.*

a) $A_n \in \mathrm{B}(\mathcal{H})$ and $\|A_n\| \leq n$;

b) $nR_n(A) + n^{-1}A_n = -1$;

c) A_n is dissipative;

d) For all $f \in D(A)$ we have $\|A_n f\|_{\mathcal{H}} \leq \|Af\|_{\mathcal{H}}$;

e) For all $f \in \mathcal{H}$ we have $\lim_{n\to\infty} \|f + nR_n(A)f\|_{\mathcal{H}} = 0$;

f) For all $f \in D(A)$ we have $\lim_{n\to\infty} \|Af - A_n f\|_{\mathcal{H}} = 0$.

Proof. a) We have

$$A_n = -(A - n)nR_n(A) - n^2 R_n(A) = -n^2 R_n(A) - n. \tag{4.27}$$

Hence $A_n \in \mathrm{B}(\mathcal{H})$ and $\|A_n\| \leq 2n$ since $\|R_n(A)\| \leq n^{-1}$ by Proposition 4.11. The improved bound $\|A_n\| \leq n$ follows from c) and Problem **??**.

b) This follows by dividing (4.27) by n.

c) By b) we have for $f \in \mathcal{H}$

$$\begin{aligned} \mathrm{Re}\langle f, A_n f\rangle &= \mathrm{Re}\langle -n^{-1}A_n - nR_n(A)f, A_n f\rangle \\ &= -n^{-1}\|A_n f\|^2 + n^2 \mathrm{Re}\langle R_n(A)f, AR_n(A)f\rangle \leq 0. \end{aligned} \tag{4.28}$$

d) Since $\|nR_n(A)\| \leq 1$ we have for $f \in D(A)$ (since $R_n(A)$ is both left and right inverse of $A - n$)

$$\|A_n f\| \overset{(4.27)}{=} \|n^2 R_n(A) f + nf\| = \|nR_n(A)Af\| \leq \|Af\|. \tag{4.29}$$

e) Let first $f \in D(A)$. Then by b)

$$\|f + nR_n(A)f\| = n^{-1}\|A_n f\| \leq n^{-1}\|Af\| \overset{n\to\infty}{\to} 0. \tag{4.30}$$

For the general case $f \in \mathcal{H}$ let $\varepsilon > 0$ and choose $g \in D(A)$ with $\|f - g\| < \varepsilon$ (using density of $D(A)$). Then

$$\|f + nR_n(A)f\| \leq \|g + nR_n(A)g\| + \|(1 + nR_n(A))(f - g)\| \leq n^{-1}\|Ag\| + 2\varepsilon, \tag{4.31}$$

so choosing n large enough this is less than 3ε, which proves the claimed convergence.

f) We have

$$\|Af - A_n f\| = \|(1 + nR_n(A))Af\|, \tag{4.32}$$

so the claim follows from e). □

Definition 4.15. Let A, $D(A)$ be densely defined. The graph norm on $D(A)$ is

$$\|f\|_{D(A)} = \sqrt{\|f\|^2 + \|Af\|^2}. \tag{4.33}$$

Remark 4.16. The graph norm is the same as the norm on $\mathcal{H} \oplus \mathcal{H}$ restricted to the graph of A:

$$\|f\|^2_{D(A)} = \|(f, Af)\|^2_{\mathcal{H}\oplus\mathcal{H}} = \langle f, f\rangle_{\mathcal{H}} + \langle Af, Af\rangle_{\mathcal{H}}. \tag{4.34}$$

The normed space $D(A), \|\cdot\|_{D(A)}$ is complete if and only if A is closed.

Theorem 4.17 (Hille-Yosida)**.** *Let $A, D(A)$ be maximal dissipative. For every $u_0 \in D(A)$ there exists a unique function*

$$u \in C^1([0,\infty), \mathcal{H}) \cap C([0,\infty), D(A)) \tag{4.35}$$

satisfying (4.1)*. Moreover, the map*

$$\begin{aligned} \Phi_A : D(A) &\to C([0,\infty), \mathcal{H}) \\ u_0 &\mapsto u \end{aligned}$$

is linear and satisfies

$$\|\Phi_A u_0\|_{C([0,\infty),\mathcal{H})} \leq \|u_0\|_{\mathcal{H}}.$$

It thus extends uniquely to a continuous map $\Phi_A : \mathcal{H} \to C([0,\infty), \mathcal{H})$ of norm one.

Proof. **Step 1 (uniqueness)** Assume u, $\tilde{u}$ are two solutions to (4.1). Then $u(0)-\tilde{u}(0) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u-\tilde{u}\|^2 = 2\mathrm{Re}\langle u-\tilde{u}, A(u-\tilde{u})\rangle \le 0. \tag{4.36}$$

Thus $t \mapsto \|u(t)-\tilde{u}(t)\|$ is non-increasing, so

$$0 \le \|u(t)-\tilde{u}(t)\| \le \|u(0)-\tilde{u}(0)\| = 0$$

and the solution is unique.

Step 2 (approximate solutions u_n)

Let $u_n(t) = \Phi_{A_n} u(t) = \mathrm{e}^{A_n t} u_0$ be the unique solution of

$$\begin{cases} \dfrac{\mathrm{d}u_n}{\mathrm{d}t} = A_n u_n\,, & t > 0 \\ u_n(0) = u_0 \end{cases} \tag{4.37}$$

(cf. Thm. 4.2).

First note that, since A_n is dissipative, we have for all $n \in \mathbb{N}$, $t \ge 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_n(t)\|^2 = 2\mathrm{Re}\langle u_n(t), A_n u_n(t)\rangle \le 0. \tag{4.38}$$

Consequently,

$$\|u_n(t)\| \le \|u_0\|. \tag{4.39}$$

Since A_n is bounded, we may apply this reasoning with inital condition $v_0 = Au_0$ and obtain with Lemma 4.14d)

$$\|A_n u_n(t)\| = \|e^{A_n t} A_n u_0\| \le \|A_n u_0\| \le \|Au_0\|. \tag{4.40}$$

Step 3 (approximation of u) We now prove that u_n converges to some limit u uniformly on compact invervalls $[0,t_0] \subset [0,\infty)$.

Let $n, m \in \mathbb{N}$. Obviously we have $u_n(0) = u_0 = u_m(0)$, and by the fundamental theorem of calculus

$$\|u_n(t)-u_m(t)\|^2 = \int_0^t 2\mathrm{Re}\langle A_n u_n(s) - A_m u_m(s), u_n(s)-u_m(s)\rangle \mathrm{d}s. \tag{4.41}$$

Now by Lemma 4.14b), we have (for fixed s, which we drop from the notation)

$$\begin{aligned}
&\mathrm{Re}\langle A_n u_n - A_m u_m, u_n - u_m\rangle \\
&= \mathrm{Re}\langle A_n u_n - A_m u_m, -nR_n(A)u_n - n^{-1}A_n u_n + mR_m(A)u_m + m^{-1}A_m u_m\rangle \\
&= \underbrace{\mathrm{Re}\langle A(nR_n(A)u_n - mR_m(A)u_m), (nR_n(A)u_n - mR_m(A)u_m)\rangle}_{\le 0} \\
&\qquad + \mathrm{Re}\langle A_n u_n - A_m u_m, -n^{-1}A_n u_n + m^{-1}A_m u_m\rangle \\
&\le (\|A_n u_n\| + \|A_m u_m\|)(n^{-1}\|A_n u_n\| + m^{-1}\|A_m u_m\|).
\end{aligned} \tag{4.42}$$

By (4.40) we have for all $0 \leq t \leq t_0$

$$\|u_n(t) - u_m(t)\|^2 \leq \int_0^t 4\|Au_0\|^2(n^{-1} + m^{-1})\mathrm{d}s \leq 4t_0(n^{-1} + m^{-1})\|Au_0\|^2. \tag{4.43}$$

This proves that $u_n(t)$ is a Cauchy sequence for every t and thus converges to a limit $u(t) \in \mathcal{H}$. Since the bound above is uniform for $t \leq t_0$, the convergence is uniform and thus $t \mapsto u(t)$ is continuous, i.e.,

$$u \in C([0, \infty), \mathcal{H}). \tag{4.44}$$

Moreover, we have

$$\|u(t)\| = \lim_{n\to\infty} \|u_n(t)\| \leq \|u_0\|. \tag{4.45}$$

The map $u_0 \mapsto u$ is also linear since $u_0 \mapsto u_n$ is linear for every n. We have thus shown that $u_0 \mapsto \Phi_A u_0 = u$ is linear and bounded, and can thus be extended to $u_0 \in \mathcal{H}$.

Step 4 (differentiability) It remains to prove that, for $u_0 \in D(A)$, u is differentiable and a solution to (4.1). For $n \in \mathbb{N}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} u_n(t) = A_n e^{A_n t} u_0 = e^{A_n t} A_n u_0. \tag{4.46}$$

Now

$$\|e^{A_n t} A_n u_0 - \Phi_A(t) A u_0\| \leq \|e^{A_n t}(A_n u_0 - Au_0)\| + \|(\Phi_A(t) - e^{A_n t})Au_0\|. \tag{4.47}$$

The first term converges to zero uniformly in t by (4.39) and Lemma 4.14f). The second term should converge to zero aswell since $u_n \to u$. However we have only proved this for $u_0 \in D(A)$ so far, and $Au_0 \notin D(A)$, in general. To close this gap, let $\varepsilon > 0$ and $v_0 \in D(A)$ with $\|v_0 - Au_0\| < \varepsilon$. Then

$$\|(\Phi_A(t) - \Phi_{A_n}(t))Au_0\| \leq \|(\Phi_A(t) - \Phi_{A_n}(t))v_0\| + \underbrace{\|(\Phi_A(t) - \Phi_{A_n}(t))(Au_0 - v_0)\|}_{<2\varepsilon}. \tag{4.48}$$

By the convergence of solutions with $u_0 \in D(A)$ proved above, we thus have for n large enough

$$\sup_{0 \leq t \leq t_0} \|\Phi_{A_n}(t) A_n u_0 - \Phi_A(t) A u_0\| < 4\varepsilon. \tag{4.49}$$

Thus $\frac{\mathrm{d}}{\mathrm{d}t} u_n$ converges uniformly to $\Phi_A A u_0$, which must then be equal to $\frac{\mathrm{d}}{\mathrm{d}t} u$, so

$$u \in C^1([0, \infty), \mathcal{H}). \tag{4.50}$$

Step 5 (u is a solution) From what we have proved so far, we know that for $t \geq 0$

$$\lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} u_n(t) = \lim_{n\to\infty} A_n u_n(t) = -\lim_{n\to\infty} AnR_n(A) u_n(t) = \frac{\mathrm{d}}{\mathrm{d}t} u(t). \tag{4.51}$$

Using that $u_n(t)$ and $-nR_n(A)u(t)$ both converge to $u(t)$ (by Lemma 4.14e)) we have for fixed $t > 0$

$$\begin{aligned}\|nR_n(A)u_n(t) + u(t)\| &\leq \|nR_n(A)(u_n(t) - u(t))\| + \|nR_n(A)u(t) + u(t)\| \\ &\leq \underbrace{\|nR_n(A)\|}_{\leq 1} \|u_n(t) - u(t)\| + \|nR_n(A)u(t) + u(t)\|, \end{aligned} \tag{4.52}$$

which tends to zero for $n \to \infty$. We thus have

$$\begin{aligned} -nR_n(A)u_n(t) &\to u(t) \\ -AnR_n(A)u_n(t) &\to \frac{\mathrm{d}}{\mathrm{d}t}u(t). \end{aligned} \tag{4.53}$$

Since A is closed by Proposition 4.11, this implies that $u(t) \in D(A)$ and $\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t)$, i.e., u is indeed a solution to (4.1). Moreover, since Au equals the (continuous) derivative of u, we have

$$u \in C([0, \infty), D(A)). \tag{4.54}$$

This completes the proof.

□

Remark 4.18. The Theorem says that the abstract Cauchy problem (4.1) is *well posed* (in the sense of Hadamard) in $D(A)$, that is, we have

1. Existence of a solution for every initial datum,
2. Uniqueness of this solution,
3. Continuous dependence of the solution on the initial data (by boudedness of Φ_A).

It is easy to generalise this result to operators with spectrum in the half plane $\{z \in \mathbb{C} : \mathrm{Re}(z) \leq \mu\}$ satisfying

$$\mathrm{Re}\langle f, Af\rangle \leq \mu \|f\|^2. \tag{4.55}$$

for some $\mu \in \mathbb{R}$. By Proposition 4.11 then $A - \mu$ is maximal dissipative. It is clear that u is a solution to (4.1) if and only if

$$u(t) = \mathrm{e}^{\mu t}\mathrm{e}^{t(A-\mu)}u_0. \tag{4.56}$$

Corollary 4.19. *Let A, $D(A)$ be densely defined, satisfy* (4.55) *for some $\mu \in \mathbb{R}$ and assume that $A - z$ is onto for some z with* $\mathrm{Re}z > \mu$. *Then* (4.56) *is the unique solution to the Cauchy Problem* (4.1). *This solution satisfies*

$$\|u(t)\|_{\mathcal{H}} \leq \mathrm{e}^{\mu t}\|u_0\|_{\mathcal{H}} \tag{4.57}$$

Definition 4.20. Let A, $D(A)$ be maximal dissipative and $t \geq 0$. We define a bounded operator e^{tA} on $\mathcal{H}$ by

$$\mathrm{e}^{At}f := (\Phi_A f)(t),$$

i.e., $\mathrm{e}^{At}f$ is the solution to the Cauchy problem (4.1) with initial condition f evaluated at time t.

We caution that e^{At} is in general not given by the exponential series, which might not converge. Moreover, it is only defined for $t \geq 0$, since Theorem 4.17 only gives existence of solutions for positive time. It could be defined for $t \leq 0$ if $-A$ is maximal dissipative.

Corollary 4.21. *Let A, $D(A)$ be maximal dissipative and $t, s \geq 0$, then*

$$e^{(t+s)A} = e^{tA} e^{sA}. \tag{4.58}$$

Proof. This follows from uniqueness of solutions as in Corollary 4.3. □

The existence result we have proved is actually optimal, in the following sense.

Theorem 4.22 (Hille-Yosida, part two)**.** *Assume that the family of bounded operators $T(t)$, $t \geq 0$, forms a strongly continuous semi-group, that is, we have*

1) $T(0) = 1$,

2) $T(t+s) = T(t)T(s)$,

3) $\forall f \in \mathcal{H} : \lim_{t \to 0} T(t)f = f$.

Assume moreover that $\|T(t)\| \leq 1$ for all $t \geq 0$. Then there exists a unique maximal dissipative operator $A, D(A)$ so that $T(t) = e^{tA}$.

Proof. We define

$$\begin{aligned} D(A) &= \Big\{ f \in \mathcal{H} : \lim_{t \to 0} t^{-1}(T(t) - 1)f \text{ exists} \Big\}, \\ Af &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} T(t)f. \end{aligned} \tag{4.59}$$

The task is then tow show that $D(A)$ is dense, A is maximal dissipative, and $e^{tA} = T(t)$.

To show density of $D(A)$, one may set for $s > 0$, $f \in \mathcal{H}$

$$f_s = \frac{1}{s} \int_0^s T(t) f \mathrm{d}t. \tag{4.60}$$

We have

$$\|f_s - f\| = \left\| \frac{1}{s} \int_0^s (T(t) - 1) f \mathrm{d}t \right\| \leq \frac{1}{s} \int_0^s \|(T(t) - 1)f\| \mathrm{d}t, \tag{4.61}$$

which tends to zero, since the integrand is smaller than ε for $s \leq \delta$ by 3). Moreover, $f_s \in D(A)$ because, with the semi-group property 2),

$$t^{-1}(T(t)f_s - f_s) = \frac{1}{ts} \int_0^s T(t+r)f - T(r)f \mathrm{d}r = \frac{1}{ts} \int_s^{s+t} T(r) f \mathrm{d}r - \frac{1}{ts} \int_0^t T(r)f \ u\mathrm{d}r, \tag{4.62}$$

which converges to $s^{-1}(T(s) - 1)f$ by the argument above. Hence, $D(A)$ is dense. Again with 2), one can see that

$$\frac{\mathrm{d}}{\mathrm{d}t} T(t) f = AT(t)f = T(t)Af, \tag{4.63}$$

so $T(t)f$ solves the Cauchy problem with initial datum f. The claim thus follows once we show that A is maximal dissipative, by uniqueness of the solution.

To start with, A is dissipative since for $f \in D(A)$

$$2\mathrm{Re}\langle f, Af\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \|T(t)f\|^2 \leq 0, \tag{4.64}$$

as $\|T(t)f\| \leq \|f\|$.

Moreover, A is closed since for $f_n \to f$, $Af_n \to g$

$$\begin{aligned}\lim_{t\to 0} t^{-1}(T(t)-1)f &= \lim_{t\to 0}\lim_{n\to\infty} t^{-1}(T(t)-1)f_n \\ &= \lim_{t\to 0}\lim_{n\to\infty} t^{-1}\int_0^t T(s)Af_n \mathrm{d}s \\ &= \lim_{t\to 0} t^{-1}\int_0^t T(s)g\, \mathrm{d}s = g,\end{aligned} \tag{4.65}$$

so $f \in D(A)$, $Af = g$. To show that $1 \in \rho(A)$, we write a formula for the resolvent by a Laplace transform

$$\langle g, Rf\rangle := -\int_0^\infty \mathrm{e}^{-t}\langle g, T(t)f\rangle \mathrm{d}t. \tag{4.66}$$

The integral makes sense for $g, f \in \mathcal{H}$ because $\|T(t)\| \leq 1$, and for all $f \in D(A)$, $g \in \mathcal{H}$

$$\langle g, (A-1)Rf\rangle = -\int_0^\infty \mathrm{e}^{-t}\langle g, AT(t)f\rangle \mathrm{d}t + \int_0^\infty \mathrm{e}^{-t}\langle g, T(t)f\rangle \mathrm{d}t = \langle g, f\rangle, \tag{4.67}$$

by integration by parts. Thus the range of $(A-1)$ contains $D(A)$ and is dense. Moreover, since A is dissipative, we have for $f \in D(A)$

$$\|f\|^2 = \|(A-1)Rf\|^2 = \|ARf\|^2 + \|Rf\|^2 - 2\mathrm{Re}\langle ARf, Rf\rangle \geq \|Rf\|^2. \tag{4.68}$$

Hence, for a sequence $f_n \in D(A)$ converging to $f \in \mathcal{H}$, the sequence $g_n = Rf_n$ is Cauchy and converges to some $g \in \mathcal{H}$. Then, since $A-1$ is closed $f_n = (A-1)g_n \to f$ and $g_n \to g$, we have $g \in D(A)$ and $f = (A-1)g$. This shows that A is maximal dissipative.

In order to prove uniqueness, let B, $D(B)$ be another maximal dissipative operator with $\mathrm{e}^{tB} = \mathrm{e}^{tA}$. Then, by defintion of $D(A)$ and the fact that $\mathrm{e}^{tB}f \in C^1(\mathbb{R}_+, \mathcal{H})$ for $f \in D(B)$, we have $D(B) \subset D(A)$ and since the derivatives are equal, A extends B. But since $A - 1 : D(A) \to \mathcal{H}$ is a bijection, no proper restriction $B - 1 : D(B) \to \mathcal{H}$ can be bijective, so $A = B$. This proves the claim. □

4.3. Applications of the Hille-Yosida theorem

4.3.1. The Schrödinger equation

We will now study the Cauchy problem for the Schrödinger equation with potential

$$\begin{cases} \mathrm{i}\dfrac{\mathrm{d}u}{\mathrm{d}t} = -\Delta u + Vu\,, \qquad t > 0, \\ u(0) = u_0 \end{cases} \tag{4.69}$$

where $V : \mathbb{R}^d \to \mathbb{R}$ is a real valued potential that acts as an operator of multiplication. Since the quantity $|u(t,x)|^2$ represents the probabilty density for the position of a particle (or the configuration of a system) at time t, we additionally want that

$$\int |u(t,x)|^2 \mathrm{d}x = \|u(t)\|^2 = 1 \tag{4.70}$$

holds for all t. This is equivalent to self-adjointness of the operator $H = -\Delta + V$ on a suitable domain. For this case, we have as a Corollary to the Hille-Yosida theorem

Corollary 4.23. *Let $H, D(H)$ be self-adjoint and $u_0 \in D(H)$. Then there exists a unique solution*

$$u \in C(\mathbb{R}, D(H)) \cap C^1(\mathbb{R}, \mathcal{H}) \tag{4.71}$$

of the Cauchy problem

$$\begin{cases} \mathrm{i}\dfrac{\mathrm{d}u}{\mathrm{d}t} = Hu\,, & t > 0 \\ u(0) = u_0 \end{cases} \tag{4.72}$$

Moreover, the solution operator $\mathrm{e}^{-\mathrm{i}tH}$ *is unitary.*

Proof. Existence and uniquencess for $t \geq 0$ follows from the Hille-Yosida theorem, since $A = -\mathrm{i}H$ is maximal dissipative. For $t \leq 0$, consider $v(t) = u(-t)$, which should solve

$$\frac{\mathrm{d}v}{\mathrm{d}t}(t) = -\frac{\mathrm{d}u}{\mathrm{d}t}(-t) = \mathrm{i}Hv(t). \tag{4.73}$$

Since $A = \mathrm{i}H$ is also maximal dissipative, existence and uniqueness of v follows.

The solution operator $\mathrm{e}^{-\mathrm{i}tH} \in \mathrm{B}(\mathcal{H})$ is thus defined for every $t \in \mathbb{R}$. Moreover, for $u_0 \in D(H)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 = 2\mathrm{Re}\langle u(t), -\mathrm{i}Hu(t)\rangle = 0, \tag{4.74}$$

so

$$\|\mathrm{e}^{-\mathrm{i}tH}u_0\| = \|u_0\|. \tag{4.75}$$

The unique extension of $\mathrm{e}^{-\mathrm{i}tH} := U$ thus satisfies $U^*U = 1$. Now let $u, v \in D(H)$ and define $u(t) := (\mathrm{e}^{-\mathrm{i}tH})^* u$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t), v\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle u, \mathrm{e}^{-\mathrm{i}tH}v\rangle = \langle u, -\mathrm{e}^{-\mathrm{i}tH}\mathrm{i}Hv\rangle = \langle \mathrm{i}Hu(t), v\rangle. \tag{4.76}$$

Hence $(\mathrm{e}^{-\mathrm{i}tH})^* = \mathrm{e}^{\mathrm{i}tH}$, since $\|\mathrm{e}^{\mathrm{i}tH}u\| = \|u\|$ we have $UU^* = 1$ and $\mathrm{e}^{-\mathrm{i}tH}$ is unitary. □

Our task is thus to find hypothesis on V that ensure self-adjointness of $-\Delta + V$ on a suitable domain – usually $H^2(\mathbb{R}^d)$. The following proposition is extremely useful for studying operators that can be decomposed into a dominant part (usually the part with the highest order derivativies) which is known to be maximal dissipative (self-adjoint), and a secondary part which can be bounded by the dominant one.

Proposition 4.24. *Let A, $D(A)$ be a maximal dissipative operator and B, $D(B)$ dissipative with $D(A) \subset D(B)$. Assume that there exists $0 \leq \varepsilon < 1$ and $C > 0$ so that for all $f \in D(A)$*

$$\|Bf\| \leq \varepsilon \|Af\| + C\|f\|.$$

Then $A + B$ with domain $D(A + B) = D(A)$ is maximal dissipative.

Proof. First note that $A + B$ is defined on $D(A)$ and for $f \in D(A)$

$$\mathrm{Re}\langle f, (A + B)f\rangle = \mathrm{Re}\langle f, Af\rangle + \mathrm{Re}\langle f, Bf\rangle \leq 0, \tag{4.77}$$

since both are dissipative. Hence $A + B$ is densely defined and dissipative. By Corollary 4.12 it is now sufficent to prove that $A + B - \lambda$ is onto for some $\lambda > 0$.

Using the hypothesis, Proposition 4.11 and Exercise 10.1 we obtain

$$\|BR_\lambda(A)\| \leq \varepsilon \|AR_\lambda(A)\| + C \|R_\lambda(A)\| \leq \varepsilon + \frac{C}{\lambda}. \tag{4.78}$$

If $\varepsilon + C/\lambda < 1$, the bounded operator $1 + BR_\lambda(A)$ is thus invertible by a Neumann series. Since $A - \lambda$ is onto, then so is

$$(1 + BR_\lambda(A))(A - \lambda) = A + B - \lambda. \tag{4.79}$$

This completes the proof. □

Corollary 4.25 (Kato-Rellich)**.** *Let H, $D(H)$ be self-adjoint and K, $D(K)$ symmetric with $D(H) \subset D(K)$. Assume that there exists $0 \leq \varepsilon < 1$ and $C > 0$ so that for all $f \in D(H)$*

$$\|Kf\| \leq \varepsilon \|Hf\| + C\|f\|.$$

Then $H + K$ with domain $D(H + K) = D(H)$ is self-adjoint.

Proof. Apply Proposition 4.24 to $A = \mathrm{i}H$, $B = \mathrm{i}K$. □

Examples 4.26.

a) Let $V \in L^\infty(\mathbb{R}, \mathbb{R})$. Then the hypothesis of Corollary 4.25 are satisfied with $\varepsilon = 0$ and $C = \|V\|_\infty$. Hence $H = -\Delta + V$ is self-adjoint on $D(H) = H^2(\mathbb{R}^d)$. We can thus solve the Schrödinger equation for every bounded potential.

b) Let $d \leq 3$ and $V \in L^2(\mathbb{R}^d, \mathbb{R})$. By Sobolev's Lemma we know that $H^2(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ for $d \leq 3$. We can bound

$$\|Vf\|_{L^2(\mathbb{R}^d)} \leq \|V\|_{L^2} \|f\|_{L^\infty}. \tag{4.80}$$

Moreover, proceeding as in the proof of Sobolev's Lemma, we have for $\delta > 0$

$$\|f\|_\infty \leq (2\pi)^{-d/2} \|\hat{f}\|_{L^1} \leq (2\pi)^{-d/2} \|(1 + \delta p^2)\hat{f}\|_{L^2} \|(1 + \delta p^2)^{-1}\|_{L^2}. \tag{4.81}$$

Since

$$\|(1+\delta p^2)^{-1}\|_{L^2}^2 = \int_{\mathbb{R}^d} \frac{\mathrm{d}p}{(1+\delta p^2)^2} = \delta^{-d/2} \int_{\mathbb{R}^d} \frac{\mathrm{d}p}{(1+p^2)^2}, \tag{4.82}$$

we thus have

$$\|Vf\|_{L^2(\mathbb{R}^d)} \leq C(\delta^{1-d/4}\|-\Delta f\| + \delta^{-d/4}\|f\|_{L^2}), \tag{4.83}$$

with

$$C = (2\pi)^{-d/2}\|V\|_{L^2}\|(1+p^2)^{-1}\|_{L^2}^2. \tag{4.84}$$

Since $1 - d/4 > 0$ we can make the constant in front of $\|-\Delta f\|$ as small as we wish, and the hypothesis of Corollary 4.25 are satisfied. We can thus solve Schrödinger's equation with $V \in L^2(\mathbb{R}^d)$ if $d \leq 3$.

c) Let $d = 3$ and consider the Hamiltonian for the electron in the Hydrogen atom

$$H = -\Delta - \frac{\alpha}{|x|} \tag{4.85}$$

with $D(H) = H^2(\mathbb{R}^3)$. The Coulomb potential can be written as

$$\frac{1}{|x|} = \underbrace{\frac{1(|x| \leq 1)}{|x|}}_{\in L^2(\mathbb{R}^3)} + \underbrace{\frac{1(|x| > 1)}{|x|}}_{\in L^\infty(\mathbb{R}^3)}. \tag{4.86}$$

By example b) above,

$$H_1 := -\Delta - \alpha\frac{1(|x| \leq 1)}{|x|} \tag{4.87}$$

is self-adjoint on $H^2(\mathbb{R}^3)$. By example a) above, after adding a bounded potential we still have a self-adjoint operator with the same domain, so H is self-adjoint.

4.3.2. The heat equation

We will now apply the Hille Yosida theorem to the heat equation with variable coefficients.

$$\begin{cases} \partial_t u(t,x) = \operatorname{div} M(x)\nabla u(t,x) \\ \quad u(0,x) = u_0(x) \end{cases} \tag{4.88}$$

This equation cannot be solved directly via the Fourier transform if the coefficient matrix M is not constant. This matrix M models material properties of the medium through which the heat is flowing, such as regions of better conductivity or different conductivity depending on the direction.

Existence of solutions follows from Theorem 4.17 and Theorem 3.17.

Corollary 4.27. *Let $M \in C^1(\mathbb{R}^d, \mathrm{B}(\mathbb{C}^d))$ be bounded with bounded derivatives and satisfy for $a > 0$*

$$\forall v \in \mathbb{C}^d, x \in \mathbb{R}^d : \langle v, M(x)v\rangle \geq a\|v\|^2. \tag{4.89}$$

Let $u_0 \in H^2(\mathbb{R}^d)$. The equation

$$\begin{cases} \partial_t u = \operatorname{div} M(x)\nabla u & t \geq 0 \\ u(0,x) = u_0(x) \end{cases} \tag{4.90}$$

has a unique solution

$$u \in C^1\Big([0,\infty), L^2(\mathbb{R}^d)\Big) \cap C\Big([0,\infty), H^2(\mathbb{R}^d)\Big). \tag{4.91}$$

Proof. We need to show that the operator $A = \operatorname{div} M\nabla$, $D(A) = H^2(\mathbb{R}^d)$ satisfies the hypothesis of Theorem 4.17, i.e. A is maximal dissipative. First, A is dissipative, since

$$\langle f, \operatorname{div} M\nabla f\rangle = -\int_{\mathbb{R}^d} \langle M(x)\nabla f(x), \nabla f(x)\rangle \mathrm{d}x \leq -a \int |\nabla f(x)|^2 \mathrm{d}x < 0. \tag{4.92}$$

To show that A is maximal disspipative, we need to prove that $A - 1$ is onto. This follows from Theorem 3.17, as this states that the equation

$$(\lambda - A)u = f \tag{4.93}$$

has a unique solution $u \in H^2(\mathbb{R}^d) = D(A)$ for all $f \in L^2(\mathbb{R}^d)$. □

While this first result is a direct corollary of the Hille-Yosida theorem, we can obtain stronger statetements using more specific properties of the equation. The heat equation belongs to the class of PDEs called *parabolic*. The most important feature of these is that the solution is more regular at positive times than at $t = 0$, i.e., they are smoothing. Apart from generalized (linear) heat equations the class of parabolic equations also contains the non-linear Ricci flow that was used in Perelman's proof of the Poincaré conjecture. The smoothing property gives rise to a preferred direction of time and a form of irreversible behaviour.

The abstract property giving rise to this behaviour is that the generator A, $D(A)$ in the Cauchy problem is

1. self-adjoint,
2. non-positive (dissipative).

Together, these imply that $\sigma(A) \subset (-\infty, 0]$. If we want to solve the equation for negative times, we can consider $v(t) = u(-t)$ for $t > 0$. This would solve the equation

$$\frac{\mathrm{d}v}{\mathrm{d}t}(t) = -\frac{\mathrm{d}u}{\mathrm{d}t}(-t) = -Av \tag{4.94}$$

Note that if $\sigma(A)$ is unbounded, then $-A$ cannot satisfy the hypothesis of corollary 4.19 for any $\mu > 0$. We thus cannot say that the evolution exists for negative times, which is in accordance with the idea that the heat flow is irreversible.

The following theorem makes precise the smoothing for a abstract form of "heat equation".

Proposition 4.28. *Let A, $D(A)$ be a self-adjoint and non-positive operator on $\mathcal{H}$. Let $u_0 \in \mathcal{H}$ and define*

$$u(t) := \mathrm{e}^{At} u_0 \in C([0,\infty), \mathcal{H}).$$

Then

$$u \in C((0,\infty), D(A)) \cap C^1((0,\infty), \mathcal{H}) \tag{4.95}$$

and u solves the equation (4.1) *for $t > 0$.*

Proof. The strategy is to assume first that $u_0 \in D(A)$ and to prove that

$$\mathrm{e}^{At} : D(A) \to C^1([\varepsilon,\infty), \mathcal{H}) \tag{4.96}$$

is bounded with respect to the norm of $\mathcal{H}$ for any $\varepsilon > 0$. Then we can again extend it to $\mathcal{H}$ and by uniqueness of the extension this will prove the claim.

Let $u_0 \in D(A)$ and $u(t) = \mathrm{e}^{At} u_0$. Let A_n be the Yosida approximants (4.26) and $u_n(t) := \mathrm{e}^{A_n t} u_0$ be the correspondign solutions to (4.37). Note that A_n is bounded and self-adjoint. Recall that $\|\mathrm{e}^{A_n t}\| \le 1$, so in particular

$$\|A_n u_n(t)\| = \|\mathrm{e}^{A_n(t-s)} A_n u_n(s)\| \le \|A_n u_n(s)\| \tag{4.97}$$

for any $0 \le s \le t$. Consequently,

$$\int_0^T t\|A_n u_n(t)\|^2 \mathrm{d}t \ge \|A_n u_n(T)\|^2 \int_0^T t\mathrm{d}t = \tfrac{1}{2}T^2 \|A_n u_n(T)\|^2. \tag{4.98}$$

Note that the right hand side is the object we want to control. On the other hand, by self-adjointness of A_n, we have

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\langle A_n u_n, u_n\rangle = \langle A_n^2 u_n, u_n\rangle + \langle A_n u_n, A_n u_n\rangle = 2\|A_n u_n\|^2. \tag{4.99}$$

After integration by parts, this gives

$$\begin{aligned}
\int_0^T t\|A_n u_n(t)\|^2 \mathrm{d}t &= \frac{1}{2}\int_0^T t\frac{\mathrm{d}}{\mathrm{d}t}\langle A_n u_n(t), u_n(t)\rangle \mathrm{d}t \\
&= \frac{T}{2}\langle A_n u_n(T), u_n(T)\rangle - \frac{1}{2}\int_0^T \langle A_n u_n(t), u_n(t)\rangle \mathrm{d}t \\
&= \tfrac{1}{2}T\langle A_n u_n(T), u_n(T)\rangle - \tfrac{1}{4}\Big(\|u_n(T)\|^2 - \|u_0\|^2\Big) \\
&\le \tfrac{1}{2}T\|A_n u_n(T)\|\|u_0\| + \tfrac{1}{4}\|u_0\|^2 \\
&\le \tfrac{1}{4}T^2\|A_n u_n(T)\|^2 + \tfrac{1}{2}\|u_0\|^2.
\end{aligned} \tag{4.100}$$

With (4.98) this gives

$$T\|\tfrac{\mathrm{d}u_n}{\mathrm{d}t}(T)\| = T\|A_n u_n(T)\| \le \sqrt{2}\|u_0\|. \tag{4.101}$$

As $\frac{\mathrm{d}u_n}{\mathrm{d}t} \to \frac{\mathrm{d}u}{\mathrm{d}t}$ (see step 4 of the proof of Theorem 4.17) and $A_n u_n \to Au$ (see step 5 of the proof of Theorem 4.17), passing to the limit $n \to \infty$ proves that

$$\sup_{t\geq\varepsilon} \|Au(t)\|_{\mathcal{H}} \leq \sqrt{2}\varepsilon^{-1}\|u_0\|_{\mathcal{H}} \tag{4.102}$$

$$\sup_{t\geq\varepsilon} \|\tfrac{\mathrm{d}u_n}{\mathrm{d}t}(t)\|_{\mathcal{H}} \leq \sqrt{2}\varepsilon^{-1}\|u_0\|_{\mathcal{H}}, \tag{4.103}$$

so Φ_A has norm less than $\sqrt{2}\varepsilon^{-1}$ as a map from $D(A) \subset \mathcal{H}$ to $C([\varepsilon,\infty), D(A))$ and $C^1([\varepsilon,\infty),\mathcal{H})$. It thus extends to $u \in \mathcal{H}$ by the B.L.T. theorem, so by choosing $\varepsilon = t/2$ we see that $\mathrm{e}^{At}u_0$ is differentiable at every $t > 0$ and the equation (4.1) holds by continuity since it holds for the approximants. □

Corollary 4.29. *Let $A = \operatorname{div} M\nabla$, $D(A) = H^2(\mathbb{R}^d)$ be as in Corollary 4.27. Let $u_0 \in L^2(\mathbb{R}^d)$, then $u(t) = \mathrm{e}^{At}u_0$ satisfies*

$$u \in C((0,\infty), H^2(\mathbb{R}^d)) \cap C^1((0,\infty), L^2(\mathbb{R}^d))$$

and u solves the heat equation (4.88) *for $t > 0$.*

Lemma 4.30. *Let A, $D(A)$ be self-adjoint on $\mathcal{H}$ and define*

$$D(A^2) := \{f \in D(A) : Af \in D(A)\}.$$

Then the operator $B = A$, $D(B) = D(A^2)$ is self-adjoint on the Hilbert space $D(A)$ with the scalar product

$$\langle f, g\rangle_{D(A)} := \langle f, g\rangle_{\mathcal{H}} + \langle Af, Ag\rangle_{\mathcal{H}}.$$

Moreover, $D(A^2)$ is dense in $\mathcal{H}$.

Proof. Let $z \in \rho(A)$ (for example $z = \pm\mathrm{i}$) and $f \in D(A)$. Then $R_z(A)f \in D(A^2)$ since

$$AR_z(A)f = R_z(A)Af \in D(A). \tag{4.104}$$

Since $D(A)$ is dense in $\mathcal{H}$ and $R_z(A) : \mathcal{H} \to D(A)$ is continuous and surjective, we can thus approximate every element of $g = R_z(A)f \in D(A)$ by $R_z(A)f_n \in D(A^2)$, i.e, $D(A^2) \subset D(A)$ is dense.

Obviously, $B = A : D(A^2) \to D(A)$ is symmetric. Moreover, the map $R_z(A) : D(A) \to D(A^2)$ is the inverse of $B - z$, so in particular $\pm\mathrm{i} \in \rho(B)$ and thus, by the criterion of [FA, Prop.4.3.4], B is self-adjoint.

As the inclusion of $(D(A), \|\cdot\|_{D(A)})$ into $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is continuous with dense range, we can also approximate every $f \in \mathcal{H}$ in $D(A^2)$. □

In view of this lemma we iteratively define

$$D(A^k) = \{f \in D(A^{k-1}) : Af \in D(A^{k-1})\}, \tag{4.105}$$

and obtain that for $\ell < k$, $D(A^k) \subset D(A^\ell)$ is dense and the restriction of A to $D(A^k)$ is self adjoint on $D(A^{k-1})$, with the convention $D(A^0) = \mathcal{H}$.

Theorem 4.31. *Let A, $D(A)$ be a self-adjoint and dissipative operator on $\mathcal{H}$. Let $u_0 \in \mathcal{H}$ and u be the solution to* (4.1) *given by Proposition 4.28. Then for all $k \in \mathbb{N}_0$*

$$u \in C^\infty((0,\infty), D(A^k)). \tag{4.106}$$

Proof. Let $t_0 > 0$. By Proposition 4.28, $u(t_0) \in D(A)$. We now consider the equation

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}s}(s) &= Bv \qquad t > 0 \\ v(0) &= u(t_0). \end{aligned} \tag{4.107}$$

on $\mathcal{H} = D(A)$, where $B = A : D(A^2) \to D(A))$ is the restriction of A to $D(A^2)$. By Lemma 4.30, B is self-adjoint and since it restricts A it is also dissipative. We may thus apply Proposition 4.28 to obtain a solution to this equation. Clearly, we have $v(t) = u(t_0 + t)$ by uniqueness of the solution of 4.28. Thus, we have that

$$u \in C((t_0,\infty), D(A^2)) \cap C^1((t_0,\infty), D(A)). \tag{4.108}$$

Then $\frac{\mathrm{d}u}{\mathrm{d}t} = Au \in C^1((t_0,\infty),\mathcal{H})$ and thus also

$$u \in C^2((t_0,\infty),\mathcal{H}), \tag{4.109}$$

for any $t_0 > 0$. This shows that

$$u \in C^\ell((0,\infty), D(A^k)) \tag{4.110}$$

for $k + \ell \leq 2$. Iterating this argument yields the same for any $k, \ell \in \mathbb{N}_0$, which proves the claim. □

Corollary 4.32. *Let $A = \operatorname{div} M\nabla$, $D(A) = H^2(\mathbb{R}^d)$ be as in Corollary 4.27 and assume additionally that $M \in C^\infty(\mathbb{R}^d, \mathrm{B}(\mathbb{C}^d))$ with bounded derivatives. Let $u_0 \in L^2(\mathbb{R}^d)$, then $u(t) = \mathrm{e}^{tA}u_0$ satisfies*

$$u \in C^\infty((0,\infty) \times \mathbb{R}^d)$$

and u is a classical solution to the heat equation (4.88) *for $t > 0$.*

Proof. In view of Theorem 4.31 it is sufficent to prove that $C^\ell(\mathbb{R}^d) \subset D(A^k)$ for some k. This will follow from Sobolev's Lemma once we prove that $D(A^k) = H^{2k}(\mathbb{R}^d)$ in the lemma below. □

Lemma 4.33. *Let $A = \operatorname{div} M\nabla$, $D(A) = H^2(\mathbb{R}^d)$ be as in Corollary 4.27 and assume additionally that $M \in C^\infty(\mathbb{R}^d, \mathrm{B}(\mathbb{C}^d)$ with bounded derivatives. Then for $k \in \mathbb{N}$*

$$D(A^k) = H^{2k}(\mathbb{R}^d). \tag{4.111}$$

Proof. Assume that $f \in D(A^2)$, that is, $f \in D(A) = H^2(\mathbb{R}^d)$ and

$$(\lambda - A)f = g \in H^2(\mathbb{R}^d), \tag{4.112}$$

for $\lambda > 0$. Then for all $\varphi \in H^2(\mathbb{R}^d)$

$$\langle g, \varphi \rangle = \langle (\lambda - A) f, \varphi \rangle = \lambda \langle f, \varphi \rangle + \langle M \nabla f, \nabla \varphi \rangle, \tag{4.113}$$

i.e., f is a weak solution to the equation $\lambda f - \operatorname{div} M \nabla f = g$. Now let $j \in \{1, \dots, d\}$, and consider

$$\begin{aligned} \langle M \nabla \partial_j f, \nabla \varphi \rangle &= \langle \partial_j M \nabla f, \nabla \varphi \rangle - \langle (\partial_j M) \nabla f, \nabla \varphi \rangle \\ &= -\langle M \nabla f, \nabla \partial_j \varphi \rangle - \langle (\partial_j M) \nabla f, \nabla \varphi \rangle. \end{aligned} \tag{4.114}$$

Choosing $\varphi \in H^3(\mathbb{R}^d)$ we can use the equation and obtain

$$\begin{aligned} \lambda \langle \partial_j f, \varphi \rangle + \langle M \nabla \partial_j f, \nabla \varphi \rangle &= - \underbrace{(\lambda \langle f, \partial_j \varphi \rangle + \langle M \nabla f, \nabla \partial_j \varphi \rangle)}_{= \langle g, \partial_j \varphi \rangle} - \langle (\partial_j M) \nabla f, \nabla \varphi \rangle \\ &= \langle \partial_j g, \varphi \rangle + \langle \operatorname{div} (\partial_j M) \nabla f, \varphi \rangle. \end{aligned} \tag{4.115}$$

Thus $\partial_j f$ is a weak solution to the equation

$$\lambda u - \operatorname{div} M \nabla u = \partial_j g + \operatorname{div} \left((\partial_j M) \nabla f \right) \in L^2(\mathbb{R}^d), \tag{4.116}$$

where the right hand side is in L^2 because $g \in D(A) = H^2$ and $f \in H^2$. By Theorem 3.17 we thus have $\partial_j f \in H^2$ and thus $f \in H^3(\mathbb{R}^d)$.

To arrive at $f \in H^4(\mathbb{R}^d)$ we consider the second derivatives. Let $i, j \in \{1, \dots, d\}$. Then by the same argument as above, $\partial_{ij} f$ is a weak solution to

$$\lambda u - \operatorname{div} M \nabla u = \partial_{ij} g + \operatorname{div} \left((\partial_{ij} M) \nabla f \right) + \operatorname{div} \left((\partial_i M) \nabla \partial_j f \right) + \operatorname{div} \left((\partial_j M) \nabla \partial_i f \right). \tag{4.117}$$

The right hand side is in L^2 since $f \in H^3$ and the derivatives of M are bounded. Hence again by Theorem 3.17 we have $\partial_{ij} f \in H^2(\mathbb{R}^d)$ and thus $f \in H^4(\mathbb{R}^d)$.

It remains to prove the claim for $k > 2$. We proceed by induction, so assume that $D(A^{k-1}) = H^{2k-2}(\mathbb{R}^d)$ holds for $k \le \ell$. For $f \in D(A^\ell) \subset D(A^{\ell-1})$ we then know that $f \in H^{2\ell-2}(\mathbb{R}^d)$. Consequently, the expression $\operatorname{div} M \nabla$ acting on $\partial_j f$ equals the operator A, and we have

$$(\lambda - A) \partial_j f = \partial_j g + \operatorname{div} \left((\partial_j M) \nabla f \right) \in H^{2\ell-4} = D(A^{\ell-2}). \tag{4.118}$$

From this we conclude that $\partial_j f \in D(A^{\ell-1}) = H^{2\ell-2}$ and thus $f \in H^{2\ell-1}(\mathbb{R}^d)$. The same reasoning for $\partial_{ij} f$ then shows that

$$(\lambda - A) \partial_{ij} f = \partial_{ij} g + \operatorname{div} \left((\partial_{ij} M) \nabla f \right) + \operatorname{div} \left((\partial_i M) \nabla \partial_j f \right) + \operatorname{div} \left((\partial_j M) \nabla \partial_i f \right), \tag{4.119}$$

where the right hand side is an element of $H^{2\ell-1-3} = D(A^{\ell-2})$. We thus have $\partial_{ij} \in D(A^{\ell-1}) = H^{2\ell-2}$ and $f \in H^{2\ell}(\mathbb{R}^d)$. This completes the proof. □

4.3.3. Inhomogeneous and time-dependent equations

A natural variation of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u = Au \tag{4.120}$$

is the inhomogeneous equation with a time-dependent source term

$$\frac{\mathrm{d}}{\mathrm{d}t}u = Au + f(t). \tag{4.121}$$

For example, in the heat equation f could model the heat flow to another system.

In the finite dimensional setting, we know that the solution to this equation is given Duhamel's, or "variation of constants", formula

$$u(t) = \mathrm{e}^{At}u_0 + \int_0^t \mathrm{e}^{(t-s)A} f(s)\mathrm{d}s. \tag{4.122}$$

We will now prove that this is also the case in our setting, and that this equation has a unique solution under appropriate hypothesis on f.

Lemma 4.34. *Let $A, D(A)$ be maximal dissipative and $f \in C(\mathbb{R}_+, \mathcal{H})$. Assume that $u_0 \in D(A)$ and $u \in C^1(\mathbb{R}_+, \mathcal{H}) \cap C(\mathbb{R}_+, D(A))$ is a solution to* (4.121)*. Then u solves the integral equation* (4.122)*.*

Proof. Consider $\mathrm{e}^{A(t-s)}u(s)$. This is differentiable in s since $\mathrm{e}^{A(t-s)}u(r)$ is continuously differentiable in r, s for $u \in C^1(\mathbb{R}_+, \mathcal{H}) \cap C(\mathbb{R}_+, D(A))$. Calculating the derivative (with the chain rule) yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{A(t-s)}u(s) = \mathrm{e}^{A(t-s)}(-A)u(s) + \mathrm{e}^{A(t-s)}(Au(s) + f(s)) = \mathrm{e}^{A(t-s)}f(s). \tag{4.123}$$

Integrating this equation gives

$$u(t) - \mathrm{e}^{At}u_0 = \int_0^t \mathrm{e}^{A(t-s)} f(s)\mathrm{d}s, \tag{4.124}$$

as claimed. □

Proposition 4.35. *Let $u_0 \in D(A)$, $f \in C^1(\mathbb{R}_+, \mathcal{H})$ and define $u \in C(\mathbb{R}_+, \mathcal{H})$ by the variation of constants formula* (4.122)*. Then $u \in C^1(\mathbb{R}_+, \mathcal{H}) \cap C(\mathbb{R}_+, D(A))$ and u solves the inhomogeneous equation* (4.121)*.*

Proof. The fact that $\mathrm{e}^{At}u_0 \in C^1(\mathbb{R}_+, \mathcal{H}) \cap C(\mathbb{R}_+, D(A))$ is part of the Hille-Yosida Theorem, so it remains to prove the same for

$$\int_0^t \mathrm{e}^{A(t-s)} f(s)\mathrm{d}s. \tag{4.125}$$

The problem is that we do not have $f \in D(A)$, so $\mathrm{e}^{A(t-s)}f(s)$ is not necessarily differentiable in t. The basic idea to solve this is to use the integral and integrate by parts, to put the derivative on f, but this needs to be properly justified. First, we write

$$\int_0^t \mathrm{e}^{A(t-s)}f(s)\mathrm{d}s = \int_0^t \mathrm{e}^{As}f(t-s)\mathrm{d}s, \tag{4.126}$$

which is an element of $C^1(\mathbb{R}_+, \mathcal{H})$, with derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{e}^{A(t-s)}f(s)\mathrm{d}s = \mathrm{e}^{At}f(0) + \int_0^t \mathrm{e}^{As}f'(t-s)\mathrm{d}s. \tag{4.127}$$

On the other hand, changing the order of integration on the set $\{(s,r) : r \le s \le t\} \subset \mathbb{R}^2$,

$$\begin{aligned}\int_0^t \mathrm{e}^{A(t-s)}f(s)\mathrm{d}s &= \int_0^t \mathrm{e}^{A(t-s)}\Big(f(0) + \int_0^s f'(r)\mathrm{d}r\Big)\mathrm{d}s \\ &= \int_0^t \mathrm{e}^{A(t-s)}f(0) + \int_0^t\int_r^t \mathrm{e}^{A(t-s)}f'(r)\mathrm{d}s\mathrm{d}r. \end{aligned}\tag{4.128}$$

Now we claim that for any $g \in \mathcal{H}$, $r \ge 0$

$$\int_r^t \mathrm{e}^{As}g\mathrm{d}s \in D(A). \tag{4.129}$$

To see this, calculate for $g \in D(A)$

$$A\int_r^t \mathrm{e}^{As}g\mathrm{d}s = \int_r^t \Big(\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{As}\Big)g\mathrm{d}s = \mathrm{e}^{At}g - \mathrm{e}^{Ar}g. \tag{4.130}$$

Now the right hand side is clearly a continuous function of g in $\mathcal{H}$, so (4.129) holds because A is closed. We thus have that (4.128) is an element of $C(\mathbb{R}_+, D(A))$. Moreover, we have

$$\begin{aligned}A\int_0^t \mathrm{e}^{A(t-s)}f(s)\mathrm{d}s &\overset{(4.128)}{=} A\int_0^t \mathrm{e}^{A\tau}f(0)\mathrm{d}\tau + \int_0^t A\int_0^{t-r} \mathrm{e}^{A\tau}f'(r)\mathrm{d}\tau\mathrm{d}r \\ &\overset{(4.130)}{=} (\mathrm{e}^{At}-1)f(0) + \int_0^t (\mathrm{e}^{A(t-r)}-1)f'(r)\mathrm{d}r \\ &= -f(t) + \mathrm{e}^{At}f(0) + \int_0^t \mathrm{e}^{A(t-r)}f'(r)\mathrm{d}r \\ &\overset{(4.127)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{e}^{A(t-s)}f(s)\mathrm{d}s - f(t). \end{aligned}\tag{4.131}$$

This proves the claim. □

The formula (4.122) is used as a starting point for the solution to many more complicated equations, like non-linear equations. For example, consider an equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}u = Au + F(t,u). \tag{4.132}$$

For example, we could have $F(u) = -\mathrm{i}|u|^2 u$ (non-linear Schrödinger equation), $F(u) = B(t)u$ (time-dependent linear equation), or $F(u) = -|u|^2 u$ (non-linear heat equation, compare Problem 28). Given a solution, we can set $f(t) = F(t, u(t))$, so the formula (4.122) will hold. Writing this formula for an arbitrary u given an integral equation for u, and one can try to solve this, for example using Banach's fixed point theorem. A solution of this equation is called a mild solution of the differential equation. The task would then be to show that this solution is differentiable and actually solves the differential equation, i.e., it is a strong solution.

A. Appendix

A.1. The Lebesgue integral

This section summarizes those results from the theory of integration that are most important for the course, see [Ru] for an introduction and [LL] for more details.

Let $\mathscr{B}(\mathbb{R}^d)$ be the Borel σ-algebra on $\mathbb{R}^d$. That is, the smallest collection of subsets $B \subset \mathbb{R}^d$ that contains all open sets and is closed under complements, finite intersections and countable unions. Elements of $\mathscr{B}$ are called measurable sets.

Definition A.1. A measure is a function

$$\mu : \mathscr{B}(\mathbb{R}^d) \to \mathbb{R}_+ \cup \{\infty\}$$

with the properties

$$\mu(\emptyset) = 0$$
$$\mu\Big(\bigcup_{j=1}^{\infty} B_j\Big) = \sum_{j=1}^{\infty} \mu(B_j)$$

for any family of disjoint sets $(B_j)_{j\in\mathbb{N}}$.

The Lebesgue measure λ is the unique measure that is invariant by translation and satisfies $\lambda([0,1]^d) = 1$.

Definition A.2. A function $f : \mathbb{R}^d \to \mathbb{C}$ is called measureable if for every $B \in \mathscr{B}(\mathbb{C}) \cong \mathscr{B}(\mathbb{R}^2)$

$$f^{-1}(B) = \{x \in \mathbb{R}^d : f(x) \in B\}$$

is measurable, i.e., an element of $\mathscr{B}(\mathbb{R}^d)$.

The characteristic function χ_B of any set $B \in \mathscr{B}(\mathbb{R}^d)$ is measurable. Its integral is defined as

$$\int \chi_B(x)\lambda(\mathrm{d}x) = \lambda(B). \tag{A.1}$$

A simple function is a linear combination of characteristic functions. Any measurable function is the pointwise limit of simple functions,

$$f(x) = \lim_{n\to\infty} \sum_{j=1}^{n} a_{j,n}\chi_{B_{j,n}}(x). \tag{A.2}$$

Moreover, if f is *non-negative*, the simple functions can be chosen so that the value in each point is increasing in n. For a non-negative function one thus defines

$$\int f(x)\lambda(\mathrm{d}x) := \lim_{n\to\infty}\sum_{j=1}^{n} a_{j,n}\lambda(B_{j,n}) \in \mathbb{R}_+ \cup \{\infty\}. \tag{A.3}$$

Since the right hand side is an increasing sequence of numbers that are positive or $+\infty$, this is well defined but possibly infinite.

Definition A.3. A positive measurable function $f : \mathbb{R}^d \to \mathbb{R}_+$ is called integrable if (A.3) is finite.

A measurable function $f : \mathbb{R}^d \to \mathbb{C}$ is called integrable if $|f|$ is integrable.

If $f : \mathbb{R}^d \to \mathbb{C}$ is integrable, then

$$\int f(x)\mathrm{d}x = \int f(x)\lambda(\mathrm{d}x) = \lim_{n\to\infty}\sum_{j=1}^{n} a_{j,n}\lambda(B_{j,n}) \tag{A.4}$$

is a well-defined complex number.

If $A \in \mathscr{B}(\mathbb{R}^d)$ is a measurable set we define

$$\int_A f(x)\mathrm{d}x = \int \chi_A(x)f(x)\mathrm{d}x, \tag{A.5}$$

where χ_A is the characteristic function. We say that f is integrable on A if $f\chi_A$ is integrable.

If f is Riemann-integrable then f is Lebesgue-integrable and the integrals are equal [Ru, Thm.11.33].

Definition A.4 (Lebesgue spaces)**.** Let $1 \le p < \infty$

$$\mathscr{L}^p(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C} : |f|^p \text{ is integrable}\}.$$

The Lebesgue space $L^p(\mathbb{R}^d)$ is the quotient of $\mathscr{L}^p(\mathbb{R}^d)$ under the equivalence relation

$$f \sim g :\Leftrightarrow \lambda\Big(\{x : f(x) \neq g(x)\}\Big) = 0$$

of equality almost everywhere. It is a Banach space with the norm

$$\|f\|_p = \left(\int |f|^p(x)\mathrm{d}x\right)^{1/p},$$

where f is any representative in the equivalence class.

For $p = \infty$ we define $\mathscr{L}^p(\mathbb{R}^d)$ as the space of measureable functions for which

$$\|f\|_\infty = \text{ess}-\text{sup}|f| := \inf\Big\{t \in \mathbb{R} : \lambda(f^{-1}(t,\infty)) = 0\Big\} \tag{A.6}$$

is finite. The Lebesgue space $L^p(\mathbb{R}^d)$ is the quotient of $\mathscr{L}^p(\mathbb{R}^d)$ by the same equivalence relation.

Proposition A.5 (Hölder's inequality). *Let $1 \le p, q \le \infty$ so that $p^{-1} + q^{-1} = 1$, with the convention that $\infty^{-1} = 0$. Then for $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ we have $fg \in L^1(\mathbb{R}^d)$ and*

$$\left| \int f(x)g(x)\mathrm{d}x \right| \le \|f\|_p \|g\|_q. \tag{A.7}$$

For $d > 1$ an important result concerns the relation of the d-dimensional integral and the iteration of lower-dimensional integrals.

Theorem A.6. *Fubini-Tonelli Let $n, m \ge 1$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ be a measurable function and $A \in \mathscr{B}(\mathbb{R}^{n+m})$.*

a) If $f \ge 0$, then

$$\int\limits_A f(x,y)\lambda(\mathrm{d}(x,y)) = \int\limits_{\pi_1(A)} \Bigg(\int\limits_{\pi_1^{-1}(\{x\})\cap A} f(x,y)\mathrm{d}y \Bigg) \mathrm{d}x = \int\limits_{\pi_2(A)} \Bigg(\int\limits_{\pi_2^{-1}(\{y\})\cap A} f(x,y)\mathrm{d}x \Bigg) \mathrm{d}y$$

where $\pi_j(A)$, $j = 1, 2$ are the projections of A to $\mathbb{R}^n$, $\mathbb{R}^m$ respectively, and the equality is understood in the sense that if one expression is infinite, all are.

b) If f is integrable on A, then

a) The functions

$$x \mapsto f(x,y), \qquad y \mapsto f(x,y)$$

are integrable on $\pi_2^{-1}(\{y\})\cap A$ for almost every $y \in \mathbb{R}^m$, respectively on $\pi_1^{-1}(\{x\})\cap A$ for almost every $x \in \mathbb{R}^n$;

b) the functions (set equal to zero where the integral is not defined)

$$\varphi(y) = \int\limits_{\pi_2^{-1}(\{y\})\cap A} f(x,y)\mathrm{d}x, \qquad \psi(x) = \int\limits_{\pi_1^{-1}(\{x\})\cap A} f(x,y)\mathrm{d}y$$

are integrable;

c) the identity

$$\int\limits_{\pi_2(A)} \varphi(y)\mathrm{d}y = \int\limits_A f(x,y)\mathrm{d}y = \int\limits_{\pi_1(A)} \psi(x)\mathrm{d}x$$

holds.

The well-known transformation formula holds for the Lebesgue integral.

Theorem A.7 (Change of variables). *Let $A \in \mathscr{B}(\mathbb{R}^d)$, let $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be a C^1-diffeomorphism, and denote by $|J(x)| := |\det D\varphi(x)|$. Then if f is integrable on A, $x \mapsto f(\varphi(x))|J(x)|$ is integrable on $\varphi^{-1}(A)$ and*

$$\int\limits_A f(x)\mathrm{d}x = \int\limits_{\varphi^{-1}(A)} f(\varphi(x))|J(x)|\mathrm{d}x.$$

The most important properties of the Lebesgue integral are the convergence theorems.

Theorem A.8 (Monotone Convergence)**.** *Let $(f_n)_n \in \mathbb{N}$ be a sequence of measurable functions with $f_n \leq f_{n+1}$ and*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

almost everywhere for some function $f : \mathbb{R}^d \to \mathbb{C}$. Then f is measureable and

$$\lim_{n\to\infty} \int f_n(x)\mathrm{d}x = \int f(x)\mathrm{d}x.$$

Theorem A.9 (Dominated Convergence)**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions and assume there is a measurable function f so that*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

almost everywhere. Assume moreover that there exists a positive, integrable function g so that

$$\forall n \in \mathbb{N} : |f_n| \leq g$$

almost everywhere. Then f is integrable and

$$\lim_{n\to\infty} \int f_n(x)\mathrm{d}x = \int f(x)\mathrm{d}x.$$

An important corollary to this result concerns the exchange of integration and differentiation.

Corollary A.10. *Let $U \subset \mathbb{R}^k$ be open and $f : U \times \mathbb{R}^d \to \mathbb{C}$ a measurable function such that*

1. *for all $\eta \in U$, $x \mapsto f(\eta, x)$ is integrable,*
2. *for almost all $x \in \mathbb{R}^d$, $\eta \mapsto f(\eta, x)$ is continuously differentiable,*
3. *there exists a positive, integrable function $g : \mathbb{R}^d \to \mathbb{R}_+$ with*

$$\forall \eta \in U : \left|\nabla_\eta f(\eta, x)\right| \leq g(x).$$

Then $\eta \mapsto \int f(\eta, x)\mathrm{d}x$ is continuously differentiable and for all $j = 1, \ldots, k$

$$\partial_{\eta_j} \int f(\eta, x)\mathrm{d}x = \int \partial_{\eta_j} f(\eta, x)\mathrm{d}x.$$

B. Problems

Problem 1. Part 1
For any $n \in \mathbb{N}$, we set $f_n := \mathbf{1}_{[n,n+1]}$.

1. Show that for any $x \in \mathbb{R}_+$, $\lim_{n\to+\infty} f_n(x) = 0$
2. Show that for any $n \in \mathbb{N}$, we have $\int_{\mathbb{R}_+} f_n(x)dx = 1$

Part 2
We will show that the sequence $(f_n)_{n\in\mathbb{N}}$ does not satisfy the following property: there exist a non-negative function $g \in L^1(\mathbb{R}_+)$ such that

$$\text{a.e. } x \in \mathbb{R}_+,\ \forall n \in \mathbb{N}, \quad |f_n(x)| \leq g(x). \tag{B.1}$$

1. Show that for any $x \in \mathbb{R}_+$
$$\sup_{n\in\mathbb{N}}\{|f_n(x)|\} = 1.$$
2. Show that, if a measurable function $g : \mathbb{R}_+ \to \mathbb{R}$ satisfying (B.1), then $g \notin L^1(\mathbb{R}_+)$.

Problem 2. Let $a \in \mathbb{C}$ such that $\mathrm{Re}(a) > 0$. The goal of this exercise is to show that

$$\forall x \in \mathbb{R}^d, \quad \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}e^{-\frac{|x|^2}{2a}}dx = a^{\frac{d}{2}}e^{-\frac{a}{2}|\xi|^2} \tag{B.2}$$

Part 1
For any $x \in \mathbb{R}$, we define $h(x) := e^{-\frac{x^2}{2a}}$. We assume that $h \in \mathscr{S}(\mathbb{R}^d)$.

1. Show that $h'(x) = -\frac{x}{a}h(x)$.
2. Show that $h' \in L^1(\mathbb{R})$ and that $\widehat{h'}(\xi) = i\xi\widehat{h}(\xi)$.
3. Show that $\widehat{h}'(\xi) = -i\widehat{xh}(\xi)$.
4. Recall that
$$\int_{\mathbb{R}} h(x)dx = \sqrt{2a\pi}.$$
Show that $\widehat{h}(0) = \sqrt{a}$.
5. Deduce that $\widehat{h}$ is the solution of the following Cauchy problem
$$\begin{cases} \widehat{h}'(\xi) = -a\xi\widehat{h}(\xi) & \text{in } \mathbb{R}, \\ \widehat{h}(0) = \sqrt{a}. \end{cases} \tag{B.3}$$

6. Deduce from that, for any $\xi \in \mathbb{R}$

$$\widehat{h}(\xi) = \sqrt{a}e^{-\frac{a}{2}\xi^2}.$$

Part 2 By remarking that for any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have

$$e^{-\frac{|x|^2}{2a}} = \prod_{j=1}^{d} h(x_j),$$

show Formula (B.2).

Problem 3. Let $u_0 \in \mathscr{S}(\mathbb{R}^d)$. For any $t \geq 0$ and $\xi \in \mathbb{R}^d$, we set

$$u(t,x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t|\xi|^2} \widehat{u}_0(\xi) d\xi.$$

Part 1

1. Show that for any $(t,x) \in (0,+\infty) \times \mathbb{R}^d$, we have

$$\partial_t u(t,x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} (-|\xi|^2) e^{ix\cdot\xi} e^{-t|\xi|^2} \widehat{u}_0(\xi) d\xi.$$

2. Show that $u \in C^\infty((0,+\infty) \times \mathbb{R}^d)$.
3. Show that $\partial_t u - \Delta u = 0$ in $(0,+\infty) \times \mathbb{R}^d$.

Part 2

1. Show that $(t,x) \in (0,+\infty) \times \mathbb{R}^d$,

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t|\xi|^2} \widehat{u}_0(\xi) d\xi = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy.$$

2. Show that $\lim_{t\to 0^+} u(t,x) = u_0(x)$.
3. Deduce that for any $x \in \mathbb{R}^d$, we have $u(0,x) = u_0(x)$.

Part 3
Show that, for any $f \in \mathscr{S}(\mathbb{R}^d)$, we have

$$f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi) d\xi.$$

Problem 4. For multiindeces $\alpha, \beta \in \mathbb{N}^d$, we declare that $\beta \leq \alpha$ if $\beta_j \leq \alpha_j$ for all $j = 1, \ldots, d$. Denote by

$$\binom{\alpha}{\beta} = \prod_{j=1}^{d} \binom{\alpha_j}{\beta_j}.$$

Prove the generalised Leibniz formula for $f, g \in C^{|\alpha|}(\mathbb{R}^d)$

$$\partial^\alpha(fg) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g).$$

Problem 5. Let $u_0 \in \mathscr{S}(\mathbb{R}^d)$. For any $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ we set

$$u(t,x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{it|\xi|^2} \widehat{u}_0(\xi) d\xi.$$

a) Show that $u \in C^\infty(\mathbb{R} \times \mathbb{R}^d)$.

b) Show that u solves the Schrödinger equation

$$\begin{cases} \partial_t u + i\Delta u = 0, & \text{in } \mathbb{R} \times \mathbb{R}^d, \\ \lim_{t\to 0} u(t,x) = u_0(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{B.4}$$

Problem 6. Let u_0 and u_1 in $\mathscr{S}(\mathbb{R}^d)$. For any $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ we set

$$u(t,x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \cos(t|\xi|) \widehat{u}_0(\xi) d\xi + \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \frac{\sin(t|\xi|)}{|\xi|} \widehat{u}_1(\xi) d\xi.$$

1. Show that $u \in C^\infty(\mathbb{R} \times \mathbb{R}^d)$.

2. Show that u solves the wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & \text{in } \mathbb{R} \times \mathbb{R}^d, \\ \lim_{t\to 0} u(t,x) = u_0(x) \quad \text{and} \quad \lim_{t\to 0} \partial_t u(t,x) = u_1(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{B.5}$$

Problem 7. Let f and g in $\mathscr{S}(\mathbb{R}^d)$ and P be a polynomial function. Show the following properties

- $fg \in \mathscr{S}(\mathbb{R}^d)$,
- $Pf \in \mathscr{S}(\mathbb{R}^d)$.

Problem 8. Let $u_0 \in \mathscr{S}(\mathbb{R}^d)$ and $v \in \mathbb{R}^d$.

1. Let us set for any $(t,\xi) \in \mathbb{R} \times \mathbb{R}^d$, $\Phi(t,\xi) := e^{itv\cdot\xi} \widehat{u}_0(\xi)$.
   a) Show that for any $t \in \mathbb{R}$, $\Phi(t,\cdot) \in \mathscr{S}(\mathbb{R}^d)$.
   b) Show that the function $u : (t,x) \in \mathbb{R} \times \mathbb{R}^d \mapsto u(t,x) := \mathcal{F}^{-1}(\Phi(t,\cdot))(x)$ satisfies
   $$\begin{cases} \partial_t u - v \cdot \nabla u = 0 & \text{in } \mathbb{R} \times \mathbb{R}^d, \\ u(0,\cdot) = u_0 & \text{in } \mathbb{R}^d. \end{cases}$$

2. Using the Fourier inversion formula, find $\varphi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ such that for any $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, we have $u(t,x) = u_0(\varphi(t,x))$.

3. Let $p \in [1, +\infty]$. Show that

$$\forall t \in \mathbb{R}, \quad \|u(t,\cdot)\|_{L^p} = \|u_0\|_{L^p}.$$

Problem 9. Let p, q and r in $[1, \infty]$ such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$. The goal of this exercise is to show that

$$\|fg\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}. \tag{B.6}$$

1. Show (B.6) for $r = \infty$.
2. Assume that $r \neq \infty$. Deduce (B.6) from the standard Hölder estimate (which correspond to the case $r = 1$).
   Hint: use that $r/p + r/q = 1$.

Problem 10. Let p, q and r in $[1, \infty]$ such that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$. The goal of this exercise is to show that $f \star g \in L^r(\mathbb{R}^d)$, with

$$\|f \star g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}. \tag{B.7}$$

1. Assume that $r = \infty$. Show that (B.7) holds.
2. Assume that $p = q = 1$. Show that (B.7) holds.
3. Assume that $p = 1$.
   a) Show that

   $$\left(\int_{\mathbb{R}^d} |f(x-y)||g(y)|dy\right)^q \leq \left(|f| \star |g|^q\right)(x)\|f\|_{L^1}^{q-1}.$$

   Hint: Remark that $|f(x-y)||g(y)| = |f(x-y)|^{1-\frac{1}{q}}|f(x-y)|^{\frac{1}{q}}|g(y)|$.

   b) Deduce from 2. that

   $$\|f \star g\|_{L^q} \leq \|f\|_{L^1}\|g\|_{L^q}.$$

4. Assume that p, q and r belong to $]1, \infty[$.
   a) Let p_1, p_2 and p_3 in $[1, \infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ and $u \in L^{p_1}(\mathbb{R}^d)$, $v \in L^{p_2}(\mathbb{R}^d)$ and $w \in L^{p_3}(\mathbb{R}^d)$. Show that

   $$\|uvw\|_{L^1} \leq \|u\|_{L^{p_1}}\|v\|_{L^{p_2}}\|w\|_{L^{p_3}}.$$

   b) Show that $|f(x-y)||g(y)| = |f(x-y)|^{p/r}|g(y)|^{q/r}|f(x-y)|^{1-p/r}|g(y)|^{1-q/r}$.

   c) Conclude.

Problem 11. Let $p \in [1,\infty]$ and $g, f \in L^p(\mathbb{R}^d)$. The goal is to show that

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}. \tag{B.8}$$

1. Show (B.8) for $p=\infty$ and $p=1$.
2. Assume that $p \in]1,\infty[$.
   a) Show that
   $$|f(x)+g(x)|^p \leq |f(x)||f(x)+g(x)|^{p-1} + |g(x)||f(x)+g(x)|^{p-1}.$$
   b) Show that
   $$\int_{\mathbb{R}^d} |f(x)||f(x)+g(x)|^{p-1}dx \leq \|f\|_{L^p}\|f+g\|_{L^p}^{\frac{p-1}{p}}.$$
   c) Deduce (B.8).

Problem 12. Let $p \in [1,\infty[$. Show that

$$\forall \lambda > 0, \quad \int_{\mathbb{R}^d} \mathbf{1}_{\{|f|\geq\lambda\}}dx \leq \frac{1}{\lambda^p}\|f\|_{L^p}^p.$$

Problem 13. Let p and q in $[1,\infty]$ such that $p<q$. Show that if $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, then $f \in L^r(\mathbb{R}^d)$ for every $r \in [p,q]$.
Hint: Use that if $r \in [p,q]$, then there exists $\theta \in [0,1]$ such that $1/r = \theta/p + (1-\theta)/q$ and show that $\|f\|_{L^r} \leq \|f\|_{L^p}^{\theta}\|f\|_{L^q}^{1-\theta}$.

Problem 14. Let f, g and h in $\mathscr{S}(\mathbb{R}^d)$. Show the following properties

- $f \star g = g \star f$,
- $f \star (g+h) = f\star g + f \star h$,
- $(f\star g)\star h = f \star (g \star h)$.

Problem 15. Let us consider a real valued function $u : [0,+\infty[\times\mathbb{R}^d \to \mathbb{R}$. solution of the wave equation.

$$\partial_t^2 u - \Delta u = 0 \quad \text{in }]0,+\infty[\times\mathbb{R}^d.$$

Assume that

(H1) $u \in C_b^2([0,+\infty[\times\mathbb{R}^d)$;

(H2) there exists $R>0$, such that $u(0,\cdot)$ and $\partial_t u(0,\cdot)$ vanish on $B(0,R) := \{x \in \mathbb{R}^d \mid |x| \leq R\}$.

The goal of this exercise is to show that

$$u = 0 \;\text{ in } K(R) := \{(t,x) \in [0,+\infty[\times\mathbb{R}^d \mid |x| \leq R-t\}.$$

Part 1

For any $\varepsilon \geq 0$ and $(t,x) \in [0,+\infty[\times\mathbb{R}^d$, we set

$$\varphi_\varepsilon(t,x) := R - (t + \sqrt{|x|^2+\varepsilon}).$$

1. Sow that for any $t \in [0,+\infty[$ and $s > 0$, the following quantity

$$E_s^\varepsilon(t) := \frac{1}{2}\int_{\mathbb{R}^d} e^{2s\varphi_\varepsilon(t,x)}(|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2)dx,$$

   is well-defined.

2. Assume that $\varepsilon > 0$.
   a) Show that

$$\frac{d}{dt}E_s^\varepsilon = -s\int_{\mathbb{R}^d} e^{2s\varphi_\varepsilon}(|\partial_t u|^2 + |\nabla u|^2)dx - 2s\int_{\mathbb{R}^d} e^{2s\varphi_\varepsilon}(\nabla\varphi_\varepsilon \cdot \nabla u)\partial_t u dx.$$

   b) Show that $\|\nabla\varphi_\varepsilon(t,\cdot)\|_{L^\infty} \leq 1$.
   (*Hint*: recall that $\|\nabla\varphi_\varepsilon(t,\cdot)\|_{L^\infty} = \sup_{x\in\mathbb{R}^d}\left(\sum_{j=1}^d |\partial_j\varphi_\varepsilon(t,x)|^2\right)^{1/2}$).
   c) Show that

$$-2\int_{\mathbb{R}^d} e^{2s\varphi_\varepsilon}(\nabla\varphi_\varepsilon \cdot \nabla u)\partial_t u dx \leq \int_{\mathbb{R}^d} e^{2s\varphi_\varepsilon}(|\partial_t u|^2 + |\nabla u|^2)dx.$$

   (*Hint*: use the estimate $2ab \leq a^2 + b^2$)
   d) Deduce that

$$\forall t \in [0,+\infty[, \quad E_s^\varepsilon(t) \leq E_s^\varepsilon(0).$$

3. Deduce from the dominated convergence theorem that

$$\forall t \in [0,+\infty[, \quad E_s^0(t) \leq E_s^0(0).$$

4. Deduce from 3. that

$$\forall t \in [0,+\infty[, \quad \lim_{s\to+\infty} E_s^0(t) = 0.$$

   (*Hint*: use that $\varphi_0(0,x) < 0$ when $x \in B(0,R)$ and (**H2**)).

5. Conclude that

$$\forall (t,x) \in K(R), \quad u(t,x) = 0.$$

Problem 16. (Heat equation in L^p (I)) For any $t > 0$ and $f \in \mathscr{S}(\mathbb{R}^d)$, we define the function $e^{t\Delta}f$ by

$$e^{t\Delta}f := f \star h_t,$$

where

$$\forall y \in \mathbb{R}^d, \quad h_t(y) := \frac{1}{(4\pi t)^{\frac{d}{2}}}e^{-\frac{|y|^2}{4t}}.$$

1. Let $f \in \mathscr{S}(\mathbb{R}^d)$. What is the Cauchy problem satisfied by $u : (t,x) \in]0,+\infty[\times\mathbb{R}^d \mapsto e^{t\Delta} f \in \mathbb{R}$.

2. Let $p \in [1,\infty[$ and $t > 0$. Show that for any $q \in [p,\infty[$, $e^{t\Delta}$ extends to a continuous operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ and that
$$\|e^{t\Delta}\|_{\mathrm{B}(L^p,L^q)} \le \|h_t\|_{L^{(1+1/q-1/p)^{-1}}}.$$

3. Show that for any $p \in [1,\infty[$, $q \in [p,\infty[$ and $t > 0$, we have
$$\|e^{t\Delta}\|_{\mathrm{B}(L^p,L^q)} \le \frac{1}{t^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}}.$$

4. Show that for any $f \in L^p(\mathbb{R}^d)$ with $p \in [1,\infty[$, the function $u : (t,x) \in]0,+\infty[\times\mathbb{R}^d \mapsto e^{t\Delta} f \in \mathbb{R}$ belong in $C^\infty(]0,+\infty[\times\mathbb{R}^d)$ and satisfies the heat equation.

5. Let $p \in [1,\infty[$ and $f \in L^p(\mathbb{R}^d)$. Show that $\lim_{t\to 0^+} e^{t\Delta} f = f$ in $L^p(\mathbb{R}^d)$.

Problem 17. (Schrödinger equation in L^2 (I)) For any $t \in R$ and $f \in \mathscr{S}(\mathbb{R}^d)$, we define the function
$$e^{it\Delta} f := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{it|\xi|^2} \widehat{f}(\xi) d\xi.$$

1. Show that for any $t \in \mathbb{R}$ the operator $e^{it\Delta}$ extends to an operator from $L^2(\mathbb{R}^d)$ into itself and that
$$\forall f \in L^2(\mathbb{R}^d), \quad \|e^{it\Delta} f\|_{L^2} = \|f\|_{L^2}.$$

Problem 18. (Schrödinger equation in L^2 (II)) Let t and s in $\mathbb{R}$. Show that

- $e^{i0\Delta} = \mathrm{Id}_{L^2}$.
- $e^{it\Delta} \circ e^{is\Delta} = e^{i(s+t)\Delta}$.
- $(e^{it\Delta})^* = e^{-it\Delta}$.

Problem 19. (Slowly decaying function) We define the set
$$\mathscr{R}(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C} \text{ measurable} : \exists a \in \mathbb{N},\ \langle\cdot\rangle^a f \in L^1(\mathbb{R}^d)\},$$
where $\langle\cdot\rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$. Sow that any element of $\mathscr{R}(\mathbb{R}^d)$ is a regular distribution.

Problem 20. (Leibniz rule in $\mathscr{S}'$) Let $\alpha \in \mathbb{N}^d$, $T \in \mathscr{S}'(\mathbb{R}^d)$ and $f \in \mathscr{S}(\mathbb{R}^d)$. Show that $\partial^\alpha(fT) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\beta-\alpha} T$.
(*Hint:* begin the induction by the case $|\alpha| = 1$.)

Problem 21. (Classical distribution) Show that the following maps are distributions

1. (Dirac distribution) For $a \in \mathbb{R}^d$, $\delta_a : f \in \mathscr{S}(\mathbb{R}^d) \mapsto f(a)$.

Problem 22. (Operations with distributions)

Derivative

1. Show that $(\mathbf{1}_{\mathbb{R}_+})' = \delta_0$ in $\mathscr{S}'(\mathbb{R})$.

2. Show that $(\mathrm{sgn})' = 2\delta_0$ in $\mathscr{S}'(\mathbb{R})$.

Multiplication

1. Let $T \in \mathscr{S}'(\mathbb{R}^d)$ and f be polynomial function. Show that $fT \in \mathscr{S}'(\mathbb{R}^d)$.

Convolution

1. Compute $\delta_a \star f$, with $a \in \mathbb{R}^d$ and $f \in \mathscr{S}(\mathbb{R}^d)$.

Fourier transform

1. Compute $\mathscr{F}(\delta_a)$ with $a \in \mathbb{R}^d$.

2. Show that $\mathscr{F}(1) = (2\pi)^{\frac{d}{2}}\delta_0$ in $\mathscr{S}'(\mathbb{R}^d)$.

Convergence

1. Show that $\lim_{t\to 0^+} h_t = \delta_0$ in $\mathscr{S}'(\mathbb{R}^d)$ for the heat kernel $h_t(y) := \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|y|^2}{4t}}$.

Problem 23. (Principal value)

1. (Principal value of $1/x$) Show that $\mathrm{vp}(\frac{1}{x}) : f \in \mathscr{S}(\mathbb{R}) \mapsto \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{f(x)}{x} dx$ is a tempered distribution.

2. Show that $\log(|\cdot|) \in \mathscr{S}^{'}(\mathbb{R})$ and that $(\log(|x|))' = \mathrm{vp}(\frac{1}{x})$ in $\mathscr{S}'(\mathbb{R})$.

3. Show that $x\mathrm{vp}(\frac{1}{x}) = 1$ in $\mathscr{S}'(\mathbb{R})$.

4. Show that $\mathscr{F}(\mathrm{vp}(\frac{1}{x})) = i\sqrt{2\pi}\,\mathbf{1}_{\mathbb{R}_+}$ in $\mathscr{S}'(\mathbb{R})$. (*Hint*: use that $x\mathrm{vp}(\frac{1}{x}) = 1$, $\mathscr{F}(xT) = -i\,\mathscr{F}(T)'$ and $(\mathbf{1}_{\mathbb{R}_+})' = \delta_0$)

Problem 24. (Wigner measure) We define the Wigner transform at scale $h > 0$ of a function $f \in L^2(\mathbb{R}^d)$ by the following formula

$$\forall (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d, \quad W^h[f](x,\xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{iy\cdot\xi} f\left(x - \frac{h}{2}y\right) \overline{f}\left(x + \frac{h}{2}y\right) dy.$$

Part 1 (Wigner transform)
Let $f \in L^2(\mathbb{R}^d)$ and $h > 0$.

1. Show that for any $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$, we have
$$W^h[f](x,\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\frac{y}{h}\cdot\xi} f\left(x - \frac{y}{2}\right) \overline{f}\left(x + \frac{y}{2}\right) dy.$$

2. Show that $W^h[f] \in L^\infty(\mathbb{R}^{2d})$ and that
$$\|W^h[f]\|_{L^\infty(\mathbb{R}^{2d})} \leq \frac{2^d}{h^d} \|f\|^2_{L^2(\mathbb{R}^d)}.$$

3. Show that
$$\|W^h[f]\|_{L^2(\mathbb{R}^{2d})} = \frac{1}{(2\pi h)^{\frac{d}{2}}} \|f\|^2_{L^2(\mathbb{R}^d)}$$

4. Show that
$$\int_{\mathbb{R}^d} W^h[f](x,\xi) d\xi = |f(x)|^2 \quad \text{and} \quad \int_{\mathbb{R}^d} W^h[f](x,\xi) dx = \frac{1}{(2\pi)^{\frac{d}{2}}} |\widehat{f}\left(\frac{\xi}{h}\right)|^2$$

5. Deduce that
$$\int_{\mathbb{R}^{2d}} W^h[f](x,\xi) d\xi dx = \|f\|^2_{L^2(\mathbb{R}^d)}.$$

Part 2 (Wigner distribution)

1. Let $f \in L^2(\mathbb{R}^d)$ and $h > 0$. Show that the map $a \in \mathscr{S}(\mathbb{R}^{2d}) \mapsto \int_{\mathbb{R}^{2d}} a(x,\xi) W^h[f](x,\xi) d\xi dx$ defines a tempered distribution on $\mathbb{R}^{2d}$. **In the following we denote also by $W^h[f]$ this distribution.**

Part 3 (Wigner measure)
Let $(f_h)_{h>0}$ be a bounded family of $L^2(\mathbb{R}^{2d})$. We say that $(f_h)_{h>0}$ admits a *Wigner measure* T, if $T \in \mathscr{S}'(\mathbb{R}^{2d})$ and if for any sequence of positive real numbers $(h_n)_{n\in\mathbb{N}}$ converging to 0, $W^{h_n}[f_{h_n}]$ converges to T in $\mathcal{S}'(\mathbb{R}^{2d})$.

1. Let $\psi \in C_0(\mathbb{R}^d)$. For any $a \in \mathscr{S}(\mathbb{R}^{2d})$, we define $T_\psi(a) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} a(x,0)\psi(x) dx$. Show that $T_\psi \in \mathscr{S}'(\mathbb{R}^{2d})$.

2. (Wigner measure of traveling wave) Let $\alpha \in]0,1[$, $k \in \mathbb{R}^d$ and $f \in C_0(\mathbb{R}^d)$. For any $h > 0$ and $x \in \mathbb{R}^d$, we set $f_h(x) := f(x) e^{\frac{i}{h^\alpha} k\cdot x}$. Show that $T_{|f|^2}$ is a Wigner measure of $(f_h)_{h>0}$.

3. (Wigner measure of Coherent states) Let us define the
$$\Psi_h^{\xi_0,x_0}(x) := \frac{1}{(\pi h)^{\frac{d}{4}}} e^{-\frac{|x-x_0|^2}{2h}} e^{\frac{i}{h}\xi_0\cdot x}.$$

a) Show that

$$W^h[\Psi_h^{\xi_0,x_0}](x,\xi) := \frac{1}{(\pi h)^N} e^{\frac{|x-x_0|^2+|\xi-\xi_0|^2}{h}}.$$

b) Show that $\delta_{(x_0,\xi_0)}$ is a Wigner measure of $(\Psi_h^{\xi_0,x_0})_{h>0}$.

Part 4 (From the Schrödinger equation to the transport equation)

1. Let $h>0$ and $f \in C^\infty(\mathbb{R}\times\mathbb{R}^d)\cap C^1(\mathbb{R};\mathscr{S}(\mathbb{R}^d))$ be a solution of the Schrödinger equation
$$\partial_t f - \frac{h}{2}\mathrm{i}\Delta f = 0 \ \text{ in } \mathbb{R}\times\mathbb{R}^d.$$
We admit that $W^h[f(t,\cdot)] \in \mathscr{S}(\mathbb{R}^{2d})$ for any $t\in\mathbb{R}$. Show that, for any $\xi\in\mathbb{R}^d$, the function $\rho : (t,x)\in\mathbb{R}\times\mathbb{R}^d \mapsto W^h[f(t,\cdot)](x,\xi)$ is a solution of the transport equation
$$\partial_t\rho + \xi\cdot\nabla\rho = 0 \ \text{ in } \mathbb{R}\times\mathbb{R}^d.$$

Problem 25 (The Shrödinger equation and uniqueness)**.** For any $t\in\mathbb{R}\backslash\{0\}$ and $x\in\mathbb{R}^d$, we set

$$k_t(x) := \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{\frac{i|x|^2}{4t}}.$$

1. Let $u_0\in\mathscr{S}'(\mathbb{R}^d)$. Show that $k_t\star u_0\in\mathscr{S}'(\mathbb{R}^d))$ and that $t\mapsto k_t\star u_0$ is continuous $C(\mathbb{R};\mathscr{S}'(\mathbb{R}^d))$. (*Hint*: show that for any $f\in\mathscr{S}(\mathbb{R}^d)$, $t\in\mathbb{R}\mapsto k_t\star f$ belongs to $C(\mathbb{R};\mathscr{S}'(\mathbb{R}^d))$ and extends this result by duality.)

   In the following, we set $u := k\star u_0$.

2. Show that u satisfies the Schrödinger equation in $\mathscr{S}'(\mathbb{R}\times\mathbb{R}^d)$, that is
$$(i\partial_t+\Delta)u = 0 \ \text{ in } \mathscr{S}'(\mathbb{R}\times\mathbb{R}^d).$$

3. Show that, for any $\varphi\in\mathscr{S}(\mathbb{R}\times\mathbb{R}^d)$ and $t\in\mathbb{R}\setminus\{0\}$
$$\int_0^t \langle u,(i\partial_t+\Delta)\varphi(s,\cdot)\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}ds = \langle u_0, i\varphi(0,\cdot)\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} - \langle u(t), i\varphi(t,\cdot)\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \quad \text{(B.9)}$$

4. Let $\psi\in\mathscr{S}(\mathbb{R}^d)$, $T>0$ and $\chi_T\in C_c^\infty(\mathbb{R})$ such that $\chi_T(t)=1$ for any $t\in[0,T]$. We define the function Φ by setting for any $(t,\xi)\in\mathbb{R}\times\mathbb{R}^d$, $\Phi^T(t,\xi) := e^{i(T-t)|\xi|^2}\widehat{\psi}(\xi)\chi_T(t)$.

   a) Show that $\varphi^T : (t,x)\in\mathbb{R}\times\mathbb{R}^d \mapsto \mathscr{F}^{-1}(\Phi^T(t,\xi))$ belongs to $\mathscr{S}(\mathbb{R}\times\mathbb{R}^d)$ and satisfies
$$i\partial_t\varphi^T + \Delta\varphi^T = 0 \ \text{ in } \ [0,T]\times\mathbb{R}^d.$$

b) Assume that $u_0 = 0$. Deduce from (B.9) that $u(T) = 0$ in $\mathscr{S}'(\mathbb{R}^d)$.

5. Deduce that for any $u_0 \in \mathscr{S}'(\mathbb{R}^d)$, the distribution $k \star u_0$ is the unique solution of the Schrödinger equation in $\mathscr{S}'(\mathbb{R} \times \mathbb{R}^d)$ satisfying (B.9) for any $\varphi \in \mathscr{S}(\mathbb{R} \times \mathbb{R}^d)$ and $t \in \mathbb{R} \setminus \{0\}$.

Problem 26 (Green's function for the Laplacian)**.**

1. Let $g : \mathbb{R} \to \mathbb{R}$ be the function given by $g(x) = \frac{1}{2}\mathrm{e}^{-|x|}$ for $x \in \mathbb{R}$. Show that g (more precisely the associated distribution, φ_g) is the unique solution in $\mathscr{S}'(\mathbb{R})$ to the equation
$$(1 - \Delta)\varphi = \delta_0 \,,$$
by
   a) the Fourier transform;
   b) using the distributional derivative.

2. Prove that for $f \in \mathscr{S}(\mathbb{R})$ the unique solution to the equation
$$(1 - \Delta)u = f$$
is
$$u(x) = \int g(x - y)f(y)\mathrm{d}y.$$

Problem 27 (Dispersive estimate for Schrödinger equation)**.** Let $u_0 \in L^p(\mathbb{R}^d)$ for $p \in [1, 2]$ and u be the solution of the Schrödinger equation in the sens of Exercise 1.1, Item 5. with initial data u_0.

1. Show that, if $u_0 \in L^1(\mathbb{R}^d)$, then
$$\forall t \in \mathbb{R} \setminus \{0\}, \quad \|u(t)\|_{L^\infty} \le \frac{1}{(4\pi t)^{\frac{d}{2}}}\|u_0\|_{L^1}$$

2. Show that, if $u_0 \in L^2(\mathbb{R}^d)$, then
$$\forall t \in \mathbb{R} \setminus \{0\}, \quad \|u(t)\|_{L^2} = \|u_0\|_{L^2}.$$
(Hint: use that $\mathscr{F}(k_t) = e^{-it|\cdot|^2}$ in $\mathscr{S}'(\mathbb{R}^d)$.)

3. (Bonus) Let p' a real number such that $\frac{1}{p} + \frac{1}{p'} = 1$. Show that,
$$\forall t \in \mathbb{R} \setminus \{0\}, \quad \|u(t)\|_{L^{p'}} \le \frac{1}{(4\pi|t|)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{2})}}\|u_0\|_{L^p}.$$
(*Hint*: use the Riesz-Thorin Theorem.)

Problem 28 (Nonlinear heat equation)**.** Let $u_0 \in L^d(\mathbb{R}^d)$.

Part 1: Functional setting. We define the set

$$\mathrm{K} := \left\{ u \in \mathscr{C}((0,+\infty); L^{2d}(\mathbb{R}^d)) \middle| \|u\|_K := \sup_{s>0} \{ s^{\frac{1}{4}} \|u(s,\cdot)\|_{L^{2d}} \} < \infty \right\}.$$

1. Show that K is a Banach space.

Part 2: Nonlinear estimates. For any u, v and w in K, we define

$$\forall t \geq 0, \quad \mathscr{T}(u,v,w)(t) := \int_0^t e^{(t-s)\Delta} \left(u(s,\cdot)v(s,\cdot)w(s,\cdot) \right) ds.$$

1. In this question we will show that for any u, v and w in K, we have $\mathscr{T}(u,v,w) \in \mathrm{K}$ and that there is a positive constant C such that

$$\|\mathscr{T}(u,v,w)\|_{\mathrm{K}} \leq C_1 \|u\|_{\mathrm{K}} \|v\|_{\mathrm{K}} \|w\|_{\mathrm{K}}. \tag{B.10}$$

   Let u, v and w in K.

   a) Show that

$$\forall t \geq 0, \quad \|\mathscr{T}(u,v,w)(t)\|_{L^{2d}} \leq \left(\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} ds \right) \|u(s,\cdot)v(s,\cdot)w(s,\cdot)\|_{L^{2d/3}}.$$

   b) Deduce that

$$\forall t \geq 0, \quad \|\mathscr{T}(u,v,w)(t)\|_{L^{2d}} \leq \left(\int_0^t \frac{1}{(t-s)^{\frac{1}{2}} s^{\frac{3}{4}}} ds \right) \|u\|_{\mathrm{K}} \|v\|_{\mathrm{K}} \|w\|_{\mathrm{K}}.$$

   c) Deduce that (B.10) holds.

   d) Show that $\mathscr{T}(u,v,w) \in \mathrm{K}$.

Part 3: Duhamel formula. Let $u_0 \in L^d(\mathbb{R}^d)$. For any $t \geq 0$ and $u \in \mathrm{K}$, we define

$$\Phi(u)(t) := e^{t\Delta} u_0 + \mathscr{T}(u,u,u)(t).$$

1. Show that there exists a constant C_2 such that for any $u \in \mathrm{K}$, we have

$$\|\Phi(u)\|_{\mathrm{K}} \leq C_2 \left(\|u_0\|_{L^d} + \|u\|_{\mathrm{K}}^3 \right) \tag{B.11}$$

   and that $\Phi \in \mathrm{K}$.

2. Show that there exists a constant C_3 such that for any u and v in K, we have

$$\|\Phi(u) - \Phi(v)\|_{\mathrm{K}} \leq C_3 \left(\|u\|_{\mathrm{K}}^2 + \|v\|_{\mathrm{K}}^2 + \|u\|_{\mathrm{K}} \|v\|_{\mathrm{K}} \right) \|u - v\|_{\mathrm{K}}. \tag{B.12}$$

   (*Hint*: use Estimate (B.10))

Part 4: Fixed point argument Let $\varepsilon > 0$. Assume that

$$\|u_0\|_{L^d} < \varepsilon. \tag{B.13}$$

and we introduce the set

$$\mathrm{B}(2\varepsilon) := \{u \in \mathrm{K} | \|u\|_{\mathrm{K}} < 2\varepsilon\}.$$

1. Show that $\mathrm{B}(2\varepsilon)$ is a complete metric space for the distance induce be K norm, namely $\|\cdot\|_{\mathrm{K}}$.
2. Show that there exists a constant C_4 which does not depends of ε, such that for any u and v in $\mathrm{B}(2\varepsilon)$, we have

$$\|\Phi(u)\|_{\mathrm{K}} \leq C_4(1+\varepsilon^2)\varepsilon$$

and

$$\|\Phi(u) - \Phi(v)\|_{\mathrm{K}} \leq C_4\varepsilon^2\|u - v\|_{\mathrm{K}}.$$

3. Choose $\varepsilon > 0$ small enough such that Φ is a strict contraction of $\mathrm{B}(2\varepsilon)$.
4. Deduce that Φ has a unique fixed point u. (*Hint*: use the Banach fixed point theorem.)

Remark B.1. The fixed points of Φ are called the mild solutions of the equation

$$\begin{cases} \partial_t u - \Delta u + u^3 = 0, & \text{in }]0, +\infty[\times\mathbb{R}^d, \\ u(0,\cdot) = u_0, & \text{in } \mathbb{R}^d, \end{cases} \tag{B.14}$$

for small enough initial data u_0. We can obtain the existence of mild solutions for u_0 large, but only for small times. The same method (so called "Kato method") can be used to show the existence of solutions to the Navier-Stokes equation.

Problem 29 (Density in $H^s(\mathbb{R}^d)$)**.** Let $s \in \mathbb{R}$. Show that $\mathscr{S}(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$.

Problem 30 (Weak and strong convergence)**.** Let $\mathcal{H}$ be a Hilbert space and $(f_n)_{n\in\mathbb{N}}$ a sequence of $\mathcal{H}$ that converges weakly to f in $\mathcal{H}$ and such that $(\|f_n\|_{\mathcal{H}})_{n\in\mathbb{N}}$ converges to $\|f\|_{\mathcal{H}}$. Show that $(f_n)_{n\in\mathbb{N}}$ converges strongly to f in $\mathcal{H}$.

Problem 31 (Local compact embedding)**.** Let $t < s$, $\varphi \in \mathscr{S}(\mathbb{R}^d)$. The goal of this exercise is to show that the multiplication by φ is a compact operator from $H^s(\mathbb{R}^d)$ to $H^t(\mathbb{R}^d)$.

Let $(u_n)_{n\in\mathbb{N}}$ be e sequence of $H^s(\mathbb{R}^d)$ such that $\sup_{n\in\mathbb{N}}\{\|u_n\|_{H^s}\} \leq 1$.

1. Show that, up to extraction of a subsequence, $(u_n)_{n\in\mathbb{N}}$ converging weakly in $H^s(\mathbb{R}^d)$ to an element u.
   Let us set $v_n := u_n - u$.

2. Show that there exists a constant C_1 such that

$$\sup_{n\in\mathbb{N}}\{\|\varphi v_n\|_{H^s}\} \leq C_1.$$

3. Show that for any positive real number R, we have

$$\|\varphi v_n\|_{H^t} \leq \int_{B(0,R)} (1+|\xi|^2)^t |\mathscr{F}(\varphi v_n)|^2 d\xi + \frac{C_1^2}{(1+R^2)^{s-t}}.$$

Let us consider $\varepsilon > 0$.

4. Show that there exists a positive real number R such that

$$\frac{C_1^2}{(1+R^2)^{s-t}} \leq \varepsilon.$$

5. For all $\xi \in \mathbb{R}^d$, we set $\psi_\xi := \mathscr{F}^{-1}((1+|\cdot|^2)^{-s}\mathscr{F}(\varphi)(\xi - \cdot))$. Show that, for any $\xi \in \mathbb{R}^d$, ψ_ξ belongs to $\mathscr{S}(\mathbb{R}^d)$ and that

$$\forall \xi \in \mathbb{R}^d, \quad \mathscr{F}(\varphi v_n)(\xi) = \langle \psi_\xi, v_n \rangle.$$

6. Deduce that for any $\xi \in \mathbb{R}^d$, we have $\lim_{n\to+\infty} \mathscr{F}(\varphi v_n)(\xi) = 0$.

7. Assume that there exists a positive real number $M > 0$ such that

$$\sup_{\xi\in B(0,R),\, n\in\mathbb{N}} \{|\mathscr{F}(\varphi v_n)|\} \leq M. \tag{B.15}$$

8. Conclude.
   We will now show (B.15).

9. Show that there exists a positive real number C_2 such that

$$\forall \mu \in \mathbb{R}^d, \quad |\widehat{\varphi}(\mu)| \leq \frac{C_2}{(1+|\mu|^2)^{\frac{d}{2}+|s|-1}}.$$

10. Show that for any $\xi \in B(0,R)$,

$$\int_{\mathbb{R}^d} (1+|\eta|^2)^{-s} |\widehat{\varphi}(\xi-\eta)|^2 d\eta$$
$$\leq C_1 \int_{|\eta|\leq 2R} (1+|\eta|^2)^{|s|} ds + C_2 \int_{|\eta|\geq 2R} \frac{(1+|\eta|^2)^{|s|}}{(1+|\xi-\eta|^2)^{\frac{d}{2}+|s|+1}} d\eta.$$

11. Deduce that there exists a positive real number C_3 such that

$$\forall \xi \in B(0,R), \ \int_{\mathbb{R}^d} (1+|\eta|^2)^{-s} |\widehat{\varphi}(\xi-\eta)|^2 d\eta \leq C_3(1+R^2)^{|s|+\frac{d}{2}}$$

(*Hint*: to bound $\int_{|\eta|\geq 2R}(1+|\eta|^2)^{-s}(1+|\xi-\eta|^2)^{-(\frac{d}{2}+|s|+1)}d\eta$, use that if $|\xi| \leq R$ and $|\eta| \geq 2R$, we have $|\xi - \eta| \geq \frac{|\eta|}{2}$.)

12. Deduce that (B.15) holds.

Problem 32 (Norm of the heat propagator)**.** Let $t > 0$. Show that $\|e^{t\Delta}\|_{\mathcal{B}(L^2(\mathbb{R}^d))} = 1$.

Problem 33. Let $k \in \mathbb{N}_0$, $f \in H^k(\mathbb{R}^d)$ and $g \in C^\infty(\mathbb{R}^d)$ with $\|\partial^\alpha g\|_\infty < \infty$ for every $\alpha \in \mathbb{N}_0^d$. Prove that $fg \in H^k(\mathbb{R}^d)$ and the generalised Leibniz rule holds for the derivatives of order $|\alpha| \leq k$.

Problem 34. Let $\mathcal{H}$ be a Hilbert space and $A \in \mathrm{B}(\mathcal{H})$.

1. Show that, if $\|A\|_{\mathrm{B}(H)} < 1$ is bounded, then $1 + A$ is invertible.
2. Show that $\sigma(A)$ is compact.

Problem 35 (The Lax-Milgram Theorem)**.** Let $\mathcal{H}$ be a Hilbert space and

$$\alpha : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

a sesquilinear form. Asssume that

- α is *bounded*: there exists $C > 0$ so that for all $f, g \in \mathcal{H}$
$$|\alpha(f, g)| \leq C\|f\|\|g\|;$$
- α is *coercive*: there exists $a > 0$ so that for all $f \in \mathcal{H}$
$$\alpha(f, f) \geq a\|f\|^2.$$

Prove that:

1. There exists $A \in \mathrm{B}(\mathcal{H})$ so that $\alpha(f, g) = \langle Af, g\rangle$;
2. A is bijective with bounded inverse satisfying $\|A^{-1}\| \leq a^{-1}$;
3. $g = A^{-1}f$ is the unique minimiser of
$$g \mapsto \alpha(g, g) - 2\mathrm{Re}\langle f, g\rangle.$$

Problem 36. Let $V \in L^\infty(\mathbb{R}^d, \mathbb{R})$ be non-negative.

1. Prove that for every $f \in L^2(\mathbb{R}^d)$ and $\lambda > 0$ there exists a unique $u \in H^1(\mathbb{R}^d)$ such that
$$\forall \varphi \in H^1(\mathbb{R}^d) : \langle \nabla u, \nabla \varphi\rangle + \langle (V + \lambda)u, \varphi\rangle = \langle f, \varphi\rangle,$$
that is, there is a unique weak solution to the equation
$$-\Delta u + Vu + \lambda u = f.$$
2. Prove that the weak solution $u \in H^1(\mathbb{R}^d)$ obtained in part 1) is an element of $H^2(\mathbb{R}^d)$.

Problem 37. For a (possibly unbounded) measurable function $\varphi : \mathbb{R}^d \to \mathbb{C}$ consider the linear map M_φ in $L^2(\mathbb{R}^d)$ defined by

$$\begin{aligned} \mathcal{D}(M_\varphi) &:= \{f \in L^2(\mathbb{R}^d) \,|\, \varphi f \in L^2(\mathbb{R}^d)\} \\ (M_\varphi f)(x) &:= \varphi(x) f(x)\,. \end{aligned}$$

Part 1: General properties

1. Show that $\mathcal{D}(M_\varphi)$ is dense in $L^2(\mathbb{R}^d)$.

2. Show that $(M_\varphi)^* = M_{\overline{\varphi}}$.

3. Show that M_φ is closed.

4. Show that the following property: If $\varphi \in L^\infty(\mathbb{R}^d)$ then M_φ is bounded, and

$$\|M_\varphi\| = \|\varphi\|_\infty = \sup \Big\{ t : \big|\{x \in \mathbb{R}^d : |\varphi(x)| \geq t\}\big| > 0 \Big\},$$

   where $|V|$ denotes the Lebesgue measure of a measurable subset $V \subset \mathbb{R}^d$.

Problem 38. Let X be a Banach space and $F : X \to X$ be a Lipschitz map.

1. Show that for any $x_0 \in X$, there exists $x \in C([0,+\infty[; X) \cap C^1(]0,+\infty[; X)$ such that

$$\begin{cases} x' = F(x), & \text{in }]0,+\infty[, \\ x(0) = x_0. \end{cases} \tag{B.16}$$

2. Show that, for a given initial data $x_0 \in X$, this solution is unique.

3. Let $g \in L^\infty(\mathbb{R}^d)$ and $f_0 \in L^2(\mathbb{R}^d)$. Solve the following equation

$$\begin{cases} \partial_t f = gf, & \text{in }]0,+\infty[\times\mathbb{R}^d, \\ f(0,\cdot) = f_0, & \text{in } \mathbb{R}^d, \end{cases} \tag{B.17}$$

   where $f : [0,+\infty[\times\mathbb{R}^d \to \mathbb{C}$.

Problem 39. Let $A \in \mathrm{B}(\mathbb{C}^d) = \mathbb{C}^{d\times d}$ and consider the linear autonomous ODE

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au(t).$$

Show that

$$\limsup_{t\to\infty} |u(t)| < \infty$$

holds for all solutions if and only if all eigenvalues of A have non-positive real part and the purely imaginary eigenvalues have equal algebraic and geometric multiplicity.

Give examples where the solution exhibits exponential/polynomial growth.

Problem 40. Let $v \in \mathbb{R}$ and $A := v \cdot \nabla$ with $D(A) := H^1(\mathbb{R})$.

1. Show that A is maximal dissipative.

2. Show that for $u_0 \in L^2(\mathbb{R})$

$$(\mathrm{e}^{At}u_0)(x) = u_0(x+tv).$$

Problem 41. Let $d \in \mathbb{N}$ and $A \in \mathrm{B}(\mathbb{C}^d)$ be a $d \times d$ matrix.

1. Assume there exists a unitary $U \in \mathrm{B}(\mathbb{C}^d)$ so that UAU^* is diagonal and give a necessary and sufficient condition on $\sigma(A)$ for A to be dissipative.
2. Let $d = 2$ and A be the non-trivial Jordan block
$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$
Give a necessary and sufficient condition on $\lambda \in \mathbb{C}$ for A to be dissipative.
3. Let A be as in part 2. and $\mathrm{Re}\lambda < 0$. Show that there exists a matrix S such that $B = SAS^{-1}$ is dissipative.

Problem 42 (Ornstein-Uhlenbeck semi-group)**.** Let us consider $\gamma : x \in \mathbb{R} \mapsto \frac{1}{\sqrt{2\pi}} e^{-\frac{|x|^2}{2}}$ and set
$$L^2(\gamma) := \Big\{ f : \mathbb{R} \mapsto \mathbb{R} \text{ measurable } : \int_{\mathbb{R}} |f(x)|^2 \gamma(x) dx < +\infty \Big\}.$$
The for any $t \geq 0$ and $f \in L^2(\gamma)$, we set
$$(\mathrm{U}_t f)(x) := \int_{\mathbb{R}} f(e^{-t}x + \sqrt{1 - e^{-2t}} y) \gamma(y) dy.$$

1. Show that for all $t \geq$ and $f \in L^2(\gamma)$, $\mathrm{U}_t f$ is well-defined.
2. Show that $(\mathrm{U}_t)_{t \geq 0}$ its a contraction semi-group on $L^2(\gamma)$.

   $\star$ In the following, we will denote by A is generator of $(\mathrm{U}_t)_{t \geq 0}$. $\star$

3. Let us set
$$\mathcal{A} := \Big\{ f \in C^\infty(\mathbb{R}) : \forall \alpha \in \mathbb{N}, \exists P \in \mathbb{R}[X] \text{ such that } |f^{(\alpha)}| \leq P \Big\}.$$
Show that $\mathcal{A} \subset D(A)$ and that for any $f \in \mathcal{A}$, we have
$$\forall x \in \mathbb{R}, \quad (Af)(x) = \partial_x^2 f(x) - x \partial_x f(x).$$
4. We admit that $\mathcal{A}$ is stable by $(\lambda - A)^{-1}$ for some $\lambda > 0$. Show that $\overline{\mathcal{A}}^{D(A)} = D(A)$ (*Hint:* use that $\mathcal{A}$ is dense in $L^2(\gamma)$).
5. Show that A is self-adjoint.

Problem 43 (The wave equation)**.** In this exercise we solve the wave equation on $\mathbb{R}^d$ using the Hille Yosida theorem. The wave equation is
$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(0) = u_0 \\ \partial_t u(0) = v_0. \end{cases} \tag{W}$$

1. Let $\mathcal{H} := H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ and let A be the operator

$$A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$$

with domain $D(A) := H^2(\mathbb{R}^d) \oplus H^1(\mathbb{R}^d)$. Show that if $(u, v) \in C^1(\mathbb{R}, \mathcal{H})$ is a solution to the Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(u, v) = A(u, v) \\ (u, v)(0) = (u_0, v_0) \end{cases} \tag{A}$$

then u solves the wave equation (W).

2. Show that (u, v) solves (A) if and only if $(\tilde{u}, \tilde{v}) = \mathrm{e}^{-t}(u, v)$ solves

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{u}, \tilde{v}) = (A - 1)(\tilde{u}, \tilde{v}) \\ (\tilde{u}, \tilde{v})(0) = (u_0, v_0). \end{cases}$$

3. Show that $A - 1$ is maximal dissipative.

4. State the existence and uniqueness result for the wave equation implied by 1. and 2. and the Hille-Yosida theorem, specifying the functional space for the solution u.

Problem 44. Let $m > 0$, $f \in \mathscr{S}(\mathbb{R})$ and set

$$u(t, x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \cos(t\sqrt{k^2 + m^2}) \mathrm{e}^{\mathrm{i}kx} \hat{f}(k) \mathrm{d}k.$$

a) Show that for all $t \in \mathbb{R}$, $(t, x) \mapsto u(t, x) \in C^2(\mathbb{R}^2)$.

b) Show that u solves the Klein–Gordon equation

$$\partial_t^2 u(t, x) = (\partial_x^2 - m^2) u(t, x)$$

with intial data

$$u(0, x) = f(x), \qquad \partial_t u(0, x) = 0.$$

c) Show that for all $t \in \mathbb{R}$, $\int |u(t, x)|^2 \mathrm{d}x \leq \|f\|^2_{L^2(\mathbb{R})}$.

Problem 45. Define for $f \in \mathscr{S}(\mathbb{R})$

$$\varphi(f) = \frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=0} x f(x).$$

a) Show that φ defines a tempered distribution.

b) Calculate the Fourier transform of φ

c) Show that $\varphi \in H^{-1}(\mathbb{R})$.

Problem 46. Let $a \in C^1(\mathbb{R}, \mathbb{R})$ satisfy $a(x) \geq 1$ for all $x \in \mathbb{R}$ and $a, \frac{\mathrm{d}a}{\mathrm{d}x} \in L^\infty(\mathbb{R})$. Prove that for $u_0 \in H^2(\mathbb{R})$ the Cauchy problem

$$\begin{cases} \partial_t u(t,x) = \partial_x a(x) \partial_x u(t,x) + \partial_x u(t,x) \\ \qquad u(0) = u_0 \end{cases}$$

admits a unique solution

$$u \in C^1([0,\infty), L^2(\mathbb{R})) \cap C^0([0,\infty), H^2(\mathbb{R})).$$

C. Notation

Symbol	Explanation	Page
$\mathbb{N}$	Natural numbers (not including zero!)	
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$	
D	Differential of a vector-valued function	
grad	Gradient of a scalar function, $\operatorname{grad} f = Df$	
div	Divergence of a vector field, $\operatorname{div} v = \operatorname{Tr}(Dv)$	
$B(x, r)$	Open ball of radius r around x	
$\mathscr{S}(\mathbb{R}^d)$	Space of Schwartz functions on $\mathbb{R}^d$	5
$\mathscr{S}'(\mathbb{R}^d)$	Space of tempered distributions on $\mathbb{R}^d$	17
$L^p(\mathbb{R}^d)$	Lebesgue space of p-integrable functions	57
$H^k(\mathbb{R}^d)$	Sobolev space of functions in $L^2(\mathbb{R}^d)$ with k weak derivatives in L^2	21
X	Usually a complex Banach space	
$\mathrm{B}(X, Y)$	Banach space of bounded linear operators from X to Y	
$\mathrm{B}(X)$	Banach space of bounded linear operators from X to X	
X'	Space of continuous linear functionals on X (=$\mathrm{B}(X, \mathbb{C})$)	16
$\mathcal{H}$	Complex (separable) Hilbert space	
$A, D(A)$	Densely defined linear operator	35
$\mathscr{G}(A)$	Graph of A	36
$\overline{A}$	Closure of $(A, D(A))$	36
$\lVert\cdot\rVert_{D(A)}$	Graph norm on $D(A)$	39
A^*	(Hilbert) adjoint of $(A, D(A))$	35
$\ker(A)$	Kernel of A	
$\operatorname{ran}(A)$	Range of A	
$\rho(A)$	Resolvent set of A	36
$R_z(A)$	Resolvent of A in $z \in \rho(A)$, $(A - z)^{-1}$	36
$\sigma(A)$	Spectrum of A	36
$C^k(U)$	Space of k-times continuously differentiable functions $U \to \mathbb{C}$	
$C_0^k(U)$	Space of k-times continuously differentiable functions $U \to \mathbb{C}$ with compact support, $\operatorname{supp} f \Subset U$	

Bibliography

[FA] Guiseppe Dito, Functional analysis. Lecture notes, Université de Bourgogne, 2024.

[Bre] Haim Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, 2011.

[LL] Elliott H. Lieb, and Michael Loss, Analysis. American Mathematical Society, 2001.

[RS1] Michael Reed and Barry Simon, Methods of modern mathematical physics: I Functional analysis (2nd ed.). Academic press, 1980.

[RS2] Michael Reed and Barry Simon, Methods of modern mathematical physics: II Fourier analysis, self-adjointness. Academic Press, 1975.

[Ru] Walter Rudin, Principles of Mathematical Analysis (3rd ed.). McGraw Hill, 1976.