Partial Differential Equations

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[1]These lecture notes are a draft and likely to contain mistakes. Please report any typos, errors, or suggestions to `jonas.lampart@u-bourgogne.fr`. Version of January 15, 2025

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1. Introduction

A partial differential equation (PDE) is an equation whose 'unknown' is a function u, and in which (partial) derivatives of that function appear. This is similar to an ordinary differential equation (ODE) but the difference is that the unknown function

$$u : \mathbb{R}^d \to \mathbb{R} \text{ (or } \mathbb{C})$$

depends on more than one variable, $d \geq 2$, and derivatives in different directions play a role. Such equations, or systems of equations, arise in many contexts mathematics and applications in physics, engineering, and the sciences – such as electrodynamics, quantum mechanics, dynamics of weather and climate, and the description of materials.

1.1. Examples

1. The heat equation
$$\partial_t u(t, x) = \Delta_x u(t, x) \tag{1.1}$$
describes diffusion of heat in a (homogeneous, isotropic) medium.

2. Schrödinger's equation
$$\mathrm{i}\partial_t \psi(t, x) = -\Delta_x \psi(t, x) + V(x)\psi(t, x) \tag{1.2}$$
describes the wave-function of a quantum particle in an external potential V.

3. The Poisson equation
$$\Delta u(x) = \rho(x) \tag{1.3}$$
gives the electric potential generated by the (static) charge distribution ρ. Maxwell's equations give a more complete description of electrodynamics.

4. The Euler equation
$$\begin{cases} \partial_t v(t, x) + v(t, x) \cdot D_x v(t, x) + \operatorname{grad}_x p(t, x) = 0 \\ \operatorname{div}_x v(t, x) = 0 \end{cases} \tag{1.4}$$
describes the velocity field $v : \mathbb{R}^d \to \mathbb{R}^d$ and pressure $p : \mathbb{R}^d \to \mathbb{R}$ of an incompressible, inviscid fluid. Similar systems, like the Navier-Stokes equations, are used to model the dynamics of fluids and gases with different properties, e.g. water waves or atmospheric currents.

5. The Cauchy-Riemann equations

$$\begin{cases} \partial_x u(x,y) - \partial_y v(x,y) = 0 \\ \partial_y u(x,y) + \partial_x v(x,y) = 0 \end{cases} \tag{1.5}$$

are satisfied by the real and imaginary part of every holomorphic function $f = u + \mathrm{i}v : \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{C}$.

Let $\alpha \in \mathbb{N}_0^d$ be a 'multi-index' and set

$$\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}}, \tag{1.6}$$

where $|\alpha| = \sum_{j=1}^d \alpha_j$. That is, α_j is the number of partial derivatives in direction j and $|\alpha|$ is the total number of derivatives. Since for $u \in C^k(U, \mathbb{C}^n)$ the partial derivatives can be taken in any order, we can thus express the tensor $D^k u$ by

$$(D^k u)_{j_1,\dots,j_k} = \frac{\partial^k u}{\partial x_{j_k} \cdots \partial x_{j_1}} = \partial^\alpha u \tag{1.7}$$

where α_i is the number of partial derivatives taken in the i-th direction, and $|\alpha| = k$.

Note that we have the generalised Leibniz rule

$$\partial^\alpha (fg) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g), \tag{1.8}$$

where $\beta \le \alpha$ if $\beta \le \alpha$ if $\beta_j \le \alpha_j$ for all $j = 1, \dots, d$, and the binomial coefficients are generalised as

$$\binom{\alpha}{\beta} = \prod_{j=1}^d \binom{\alpha_j}{\beta_j}. \tag{1.9}$$

Definition 1.1 (Linear PDE)**.** A PDE is called (inhomogneous) linear PDE of order k if it has the form

$$\sum_{|\alpha| \le k} a_\alpha(x) \partial^\alpha u = f(x), \tag{1.10}$$

where $a_\alpha : \mathbb{R}^d \to \mathbb{C}^{n \times n}$, for $|\alpha| \le k$, and $f : \mathbb{R}^d \to \mathbb{C}^n$. The functions a_α are called the coefficients, and the PDE is called homogeneous if $f = 0$.

Question 1.2. Which of the examples in Sect. 1.1 are linear (in-) homogeneous PDEs?

2. Linear PDEs with constant coefficients and the Fourier transform

A particularly simple case of linear differential equations are those with constant coefficients, where the functions $a_\alpha(x) \equiv a_\alpha$ are independent of x. These can be transformed into simpler equations by the Fourier transform.

For $f \in L^1(\mathbb{R}^d)$, the Fourier transform is defined by

$$\hat{f}(p) = (\mathscr{F} f)(p) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}p\cdot x} f(x)\mathrm{d}x. \tag{2.1}$$

Formally, we have with $p^\alpha = \prod_{j=1}^d p_j^{\alpha_j}$

$$\begin{aligned} p^\alpha \hat{f}(p) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} p^\alpha \mathrm{e}^{-\mathrm{i}p\cdot x} f(x)\mathrm{d}x \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (-\mathrm{i})^{-|\alpha|} (\partial_x^\alpha \mathrm{e}^{-\mathrm{i}p\cdot x}) f(x)\mathrm{d}x \\ &\overset{!}{=} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (\mathrm{i})^{|\alpha|}(-1)^{|\alpha|} \mathrm{e}^{-\mathrm{i}p\cdot x} \partial_x^\alpha f(x)\mathrm{d}x \\ &= (-\mathrm{i})^{|\alpha|} \widehat{\partial_x^\alpha f}(p), \end{aligned}$$

but the integration by parts (without boundary terms!) in the penultimate step certainly needs justification.

If we accept this identity, the linear PDE of Def. 1.1 becomes after transformation

$$\Big(\sum_{|\alpha|\le k} a_\alpha (\mathrm{i}p)^\alpha \Big) \hat{u}(p) = \hat{f}(p). \tag{2.2}$$

Any solution then satisfies, formally,

$$\hat{u}(p) \overset{!}{=} \Big(\sum_{|\alpha|\le k} a_\alpha (\mathrm{i}p)^\alpha \Big)^{-1} \hat{f}(p).$$

To recover the solution u, however, we will need to invert the Fourier transform.

2.1. Basic properties

A few important properties of the Fourier transform of $f \in L^1(\mathbb{R}^d)$ follow immediately from the definition.

Proposition 2.1. *Let $f \in L^1(\mathbb{R}^d)$, denote by $\hat{f}$ its Fourier transform* (2.1) *and denote by $Rf(x) = f(-x)$ the reflection at $x = 0$ and $T_a f(x) = f(x-a)$ the translation. Then*

a) $\widehat{T_a f} = \mathrm{e}^{-\mathrm{i}ap}\hat{f}$

b) $T_a \hat{f} = \widehat{\mathrm{e}^{\mathrm{i}ax} f}$

c) $R\hat{f} = \widehat{Rf}$

d) $\overline{\hat{f}} = R\hat{\overline{f}}$

e) If f is real and even ($Rf = f$) then $\hat{f}$ is also real and even.

Proof. Properties *a)–d)* follow from simple changes of variables (exercise). Property *e)* follows by combining *c*) and *d*). □

The Dominated Convergence Theorem A.9 also yields that $\hat{f}$ is continuous.

Lemma 2.2. *Let $f \in L^1(\mathbb{R}^d)$ and $\hat{f}$ its Fourier transform* (2.1)*, then $\hat{f}$ is continuous.*

Proof. Let $p_n \to p$ be a convergent sequence. Then since $|\mathrm{e}^{-\mathrm{i}p_n x} f(x)| \leq |f(x)| \in L^1(\mathbb{R}^d)$

$$\lim_{n\to\infty} \hat{f}(p_n) = \lim_{n\to\infty} \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{-\mathrm{i}p_n x} f(x) \mathrm{d}x = \frac{1}{(2\pi)^{d/2}} \int \lim_{n\to\infty} \mathrm{e}^{-\mathrm{i}p_n x} f(x) \mathrm{d}x = \hat{f}(p) \tag{2.3}$$

by Dominated Convergence A.9, which proves the claim. □

2.2. The Schwartz space $\mathscr{S}$

In order to make the formal calculations from the introduction rigorous and derive consequences for the solutions to the PDE, we start by introducing a class of functions on which the calculations can easily be justified. We will later expand beyond this class by approximation arguments.

A good framework to consider identities such as (2.2) is the space of Schwartz functions, where we can

- differentiate
- multiply by polynomials
- define the Fourier transform and its inverse.

Definition 2.3. The Schwartz space is

$$\mathscr{S}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) \middle| \forall \alpha, \beta \in \mathbb{N}_0^d : \sup_{x\in\mathbb{R}^d} |x^\alpha \partial_x^\beta f(x)| < \infty \right\}. \tag{2.4}$$

A sequence f_n, $n \in \mathbb{N}$ in $\mathscr{S}$ converges to $f \in \mathscr{S}$ iff

$$\forall \alpha, \beta \in \mathbb{N}_0^d : \lim_{n\to\infty} \|f_n - f\|_{\alpha,\beta} = 0, \tag{2.5}$$

where

$$\|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} |x^\alpha \partial_x^\beta f(x)|. \tag{2.6}$$

A map $T : \mathscr{S}(\mathbb{R}^d) \to X$ into a metric space X is continuous iff T is sequentially continuous, that is, if for every sequence f_n converging to $f \in \mathscr{S}(\mathbb{R}^d)$

$$\lim_{n\to\infty} Tf_n = Tf \tag{2.7}$$

converges in X.

Question 2.4. Which of the following functions are elements of $\mathscr{S}(\mathbb{R})$?

1. $x \mapsto \cos(x)$,
2. $x \mapsto \cosh(x)^{-1} = 2(e^x + e^{-x})^{-1}$,
3. $x \mapsto e^{-|x|}$,
4. $x \mapsto e^{-x^2}$.

Remark 2.5. The space $\mathscr{S}$ is a complete metric space with the distance

$$d(f,g) = \sum_{n\in\mathbb{N}_0} 2^{-n} \max_{|\alpha|+|\beta|=n} \frac{\|f-g\|_{\alpha,\beta}}{1+\|f-g\|_{\alpha,\beta}}. \tag{2.8}$$

The notion of convergence defined above is the same as the convergence in the metric d.

Remark 2.6. Functions in $\mathscr{S}$ are smooth by definition, and decrease faster than any inverse polynomial. Hence $\mathscr{S} \subset L^\infty$ with $\|f\|_\infty = \|f\|_{0,0}$, and $\mathscr{S} \subset L^p$ for any $1 \le p < \infty$, as by the multinomial formula

$$\begin{aligned} |f(x)| &\le (1+x^{2d})^{-1} \sup_{y\in\mathbb{R}^d} |(1+y^{2d})f(y)| \\ &\le (1+x^{2d})^{-1} \sup_{y\in\mathbb{R}^d} \Big|\Big(1+\sum_{|\alpha|=d} \frac{d!}{\alpha!} y^{2\alpha}\Big) f(y)\Big| \\ &\le (1+x^{2d})^{-1} (\|f\|_{0,0} + \sum_{|\alpha|=d} \frac{d!}{\alpha!} \|f\|_{2\alpha,0}), \end{aligned}$$

and

$$\int (1+x^{2d})^{-1} < \infty \tag{2.9}$$

for $p \ge 1$.

For $f, g \in \mathscr{S}(\mathbb{R}^d)$ we define the convolution by

$$(f * g)(x) := \int f(x-y)g(y)\mathrm{d}y. \tag{2.10}$$

Lemma 2.7. *Let $f, g \in \mathscr{S}(\mathbb{R}^d)$ with $\int g = 1$ and set $g_n(x) = n^d g(nx)$, then*

$$\lim_{n\to\infty} (f * g_n)(x) = f(x).$$

Proof. By a change of variable

$$\int f(x-y) n^d g(ny) \mathrm{d}y = \int f(x - n^{-1}y) g(y) \mathrm{d}y. \tag{2.11}$$

Now the integrand converges pointwise to $f(x)g(y)$ and is bounded by $\|f\|_\infty |g(y)| \in L^1$, so the integral converges to $f(x)\int g = f(x)$ by Dominated Convergence. □

With this Lemma, we can prove the Fourier inversion theorem on $\mathscr{S}$.

Proposition 2.8. *Define*

$$(\mathscr{F}^{-1} f)(x) := \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{\mathrm{i}px} f(p) \mathrm{d}p.$$

Then for all $f \in \mathscr{S}(\mathbb{R}^d)$,

$$f = \mathscr{F}^{-1}\mathscr{F} f = \mathscr{F}\mathscr{F}^{-1} f.$$

Proof. We admit that $\hat{f} \in \mathscr{S}$, which is proved in Proposition 2.9 below. Let $g(x) = \mathrm{e}^{-x^2/2}$ and $g_n(x) = g(n^{-1}x)$. Then

$$(\mathscr{F}^{-1}\hat{f}(x) = \lim_{n\to\infty} \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{\mathrm{i}px} g_n(p) f(p) \mathrm{d}p \tag{2.12}$$

by Dominated Convergence. On the other hand, by Fubini,

$$\frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{\mathrm{i}px} g_n(p) \hat{f}(p) \mathrm{d}p = \frac{1}{(2\pi)^d} \int \mathrm{e}^{\mathrm{i}px} \mathrm{e}^{-\mathrm{i}py} g_n(p) f(y) \mathrm{d}y \mathrm{d}p = \frac{(\hat{g}_n * f)(x)}{(2\pi)^{d/2}}. \tag{2.13}$$

Now $\hat{g}_n(x) = n^d \mathrm{e}^{-n^2x^2/2}$ (see Problem 2), so by the preceding lemma

$$\lim_{n\to\infty} (\hat{g}_n * f)(x) = f(x) \int g = (2\pi)^{d/2} f(x), \tag{2.14}$$

and thus $(\mathscr{F}^{-1}\hat{f})(x) = f(x)$. The proof for $\mathscr{F}(\mathscr{F}^{-1}f)(x) = f(x)$ is the same. □

Proposition 2.9. *The Fourier transform $\mathscr{F}$ is a linear and continuous map*

$$\mathscr{F} : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d), \qquad f \mapsto \hat{f}.$$

Its continuous inverse is given by $\mathscr{F}^{-1}$. Moreover, the identities

$$(\partial^\alpha \mathscr{F} f)(p) = (\mathscr{F}(-\mathrm{i}x)^\alpha f)(p) \tag{2.15}$$

$$p^\alpha (\mathscr{F} f)(p) = \mathscr{F}(\mathrm{i}^{|\alpha|}\partial^\alpha) f)(p) \tag{2.16}$$

hold for all $f \in \mathscr{S}(\mathbb{R}^d)$, $\alpha \in \mathbb{N}_0^d$.

Proof. We prove that $\hat{f}$ is smooth and the first identity by induction on $|\alpha|$. For $|\alpha| = 0$ we only need to prove that $\hat{f}$ is contiuous, which is Lemma 2.2.

Now assume the statement holds for all $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq k$ and let $|\alpha| = k+1$. Then there are $\beta \in \mathbb{N}_0^d$ and $j \in \{1, \dots, d\}$ with $\alpha = \beta + e_j$. Denote $g = \partial^\beta \hat{f}$. By the theorem on parameter-dependent integrals A.10 and the induction hypothesis

$$g(p) = \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{-\mathrm{i}px} (-\mathrm{i}x_j)^\beta f(x) \mathrm{d}x \tag{2.17}$$

is differentiable, with

$$\partial_{p_j} g(p) = \frac{1}{(2\pi)^{d/2}} \int \partial_{p_j} \mathrm{e}^{-\mathrm{i}px} (-\mathrm{i}x_j)^\beta f(x) \mathrm{d}x = \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{-\mathrm{i}px} (-\mathrm{i}x_j)^\alpha f(x) \mathrm{d}x. \tag{2.18}$$

This completes the induction.

For the second identity, we use that

$$\begin{aligned} p_j \hat{f}(p) &= \frac{1}{(2\pi)^{d/2}} \int p_j \mathrm{e}^{-\mathrm{i}px} f(x) \mathrm{d}x = \frac{1}{(2\pi)^{d/2}} \int \mathrm{i}\partial_{p_j} \mathrm{e}^{-\mathrm{i}px} f(x) \mathrm{d}x \\ &= \frac{1}{(2\pi)^{d/2}} \int -\mathrm{i}\partial_{x_j} (\mathrm{e}^{-\mathrm{i}px} f(x)) + \mathrm{i}\mathrm{e}^{-\mathrm{i}px} \partial_{x_j} f(x) \mathrm{d}x. \end{aligned} \tag{2.19}$$

The integral of the derivative vanishes, because for $h \in \mathscr{S}$ by Fubini's Theorem A.6 and the fundamental theorem of calculus

$$\begin{aligned} \int \partial_{x_j} h(x) \mathrm{d}x &= \int_{|x_j| \leq R} \partial_{x_j} h(x) \mathrm{d}x + \int_{|x_j| > R} \partial_{x_j} h(x) \mathrm{d}x \\ &= \int_{\mathbb{R}^{d-1}} h(x) \Big|_{x_j = -R}^{x_j = R} + \int_{|x_j| > R} \partial_{x_j} h(x) \mathrm{d}x, \end{aligned} \tag{2.20}$$

and

$$\lim_{R \to \infty} \int_{\mathbb{R}^{d-1}} h(x) \Big|_{x_j = -R}^{x_j = R} = 0 = \int_{|x_j| > R} \partial_{x_j} h(x) \mathrm{d}x, \tag{2.21}$$

since h vanishes faster than any polynomial. This proves the second identity in the case $|\alpha| = 1$, from which the general case follows by induction, like the first.

We have shown that $\hat{f}$ is smooth, so to show that $\hat{f} \in \mathscr{S}$ we need to show that $\|\hat{f}\|_{\alpha,\beta}$ is finite. Using the identities, we find using the Leibniz rule

$$\begin{aligned} \|\hat{f}\|_{\alpha,\beta} &= \sup_{p \in \mathbb{R}^d} \left| (\mathscr{F} \partial^\alpha x^\beta)(p) f \right| \\ &\leq \frac{1}{(2\pi)^{d/2}} \int (1 + x^{2d})^{-1} \sup_{y \in \mathbb{R}^d} (1 + y^{2d}) |\partial^\alpha y^\beta f(y)| \\ &\leq C \sum_{\substack{|\gamma| \leq \beta + 2d \\ |\delta| \leq |\alpha|}} \|f\|_{\gamma,\delta} \end{aligned} \tag{2.22}$$

for some constant C, and hence $\hat{f} \in \mathscr{S}$. Moreover, $\mathscr{F}$ is continuous in $f = 0$ by the bound (2.22), so it is continuous by linearity. Continuity of the inverse follows from $\mathscr{F}^{-1} = R\mathscr{F}$. $\square$

Corollary 2.10. *Let* $f, g \in \mathscr{S}(\mathbb{R}^d)$*, then*

$$\int f(x)\hat{g}(x) = \int \hat{f}(x)g(x),$$

and

$$\int |\widehat{f}|^2(p)\mathrm{d}p = \int |f|^2(x)\mathrm{d}x.$$

Proof. The first statement follows directly from Fubini's theorem. The second is a consequence of this and the Fourier inversion formula together with $\mathscr{F}^{-1}f(x) = \hat{f}(-x) = Rf$ and Proposition 2.1d), i.e.,

$$\int \hat{f}(p)\overline{\hat{f}}(p)\mathrm{d}p = \int f(x)\hat{\overline{\hat{f}}}(x)\mathrm{d}x = \int f(x)\widehat{R\hat{\overline{f}}}(x)\mathrm{d}x = \int |f(x)|^2\mathrm{d}x. \tag{2.23}$$

□

A. Appendix

A.1. The Lebesgue integral

This section summarizes those results from the theory of integration that are most important for the course, see [Ru] for an introduction and [LL] for more details.

Let $\mathscr{B}(\mathbb{R}^d)$ be the Borel σ-algebra on $\mathbb{R}^d$. That is, the smallest collection of subsets $B \subset \mathbb{R}^d$ that contains all open sets and is closed under complements, finite intersections and countable unions. Elements of $\mathscr{B}$ are called measurable sets.

Definition A.1. A measure is a function

$$\mu : \mathscr{B}(\mathbb{R}^d) \to \mathbb{R}_+ \cup \{\infty\}$$

with the properties

$$\mu(\emptyset) = 0$$
$$\mu\Big(\bigcup_{j=1}^{\infty} B_j\Big) = \sum_{j=1}^{\infty} \mu(B_j)$$

for any family of disjoint sets $(B_j)_{j\in\mathbb{N}}$.

The Lebesgue measure λ is the unique measure that is invariant by translation and satisfies $\lambda([0,1]^d) = 1$.

Definition A.2. A function $f : \mathbb{R}^d \to \mathbb{C}$ is called measureable if for every $B \in \mathscr{B}(\mathbb{C}) \cong \mathscr{B}(\mathbb{R}^2)$

$$f^{-1}(B) = \{x \in \mathbb{R}^d : f(x) \in B\}$$

is measurable, i.e., an element of $\mathscr{B}(\mathbb{R}^d)$.

The characteristic function χ_B of any set $B \in \mathscr{B}(\mathbb{R}^d)$ is measurable. Its integral is defined as

$$\int \chi_B(x)\lambda(\mathrm{d}x) = \lambda(B). \tag{A.1}$$

A simple function is a linear combination of characteristic functions. Any measurable function is the pointwise limit of simple functions,

$$f(x) = \lim_{n\to\infty} \sum_{j=1}^{n} a_{j,n}\chi_{B_{j,n}}(x). \tag{A.2}$$

Moreover, if f is *non-negative*, the simple functions can be chosen so that the value in each point is increasing in n. For a non-negative function one thus defines

$$\int f(x)\lambda(\mathrm{d}x) := \lim_{n\to\infty} \sum_{j=1}^{n} a_{j,n}\lambda(B_{j,n}) \in \mathbb{R}_+ \cup \{\infty\}. \tag{A.3}$$

Since the right hand side is an increasing sequence of numbers that are positive or $+\infty$, this is well defined but possibly infinite.

Definition A.3. A positive measurable function $f : \mathbb{R}^d \to \mathbb{R}_+$ is called integrable if (A.3) is finite.

A measurable function $f : \mathbb{R}^d \to \mathbb{C}$ is called integrable if $|f|$ is integrable.

If $f : \mathbb{R}^d \to \mathbb{C}$ is integrable, then

$$\int f(x)\mathrm{d}x = \int f(x)\lambda(\mathrm{d}x) = \lim_{n\to\infty} \sum_{j=1}^{n} a_{j,n}\lambda(B_{j,n}) \tag{A.4}$$

is a well-defined complex number.

If $A \in \mathscr{B}(\mathbb{R}^d)$ is a measurable set we define

$$\int_A f(x)\mathrm{d}x = \int \chi_A(x) f(x)\mathrm{d}x, \tag{A.5}$$

where χ_A is the characteristic function. We say that f is integrable on A if $f\chi_A$ is integrable.

If f is Riemann-integrable then f is Lebesgue-integrable and the integrals are equal [Ru, Thm.11.33].

Definition A.4 (Lebesgue spaces)**.** Let $1 \le p < \infty$

$$\mathscr{L}^p(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C} : |f|^p \text{ is integrable}\}.$$

The Lebesgue space $L^p(\mathbb{R}^d)$ is the quotient of $\mathscr{L}^p(\mathbb{R}^d)$ under the equivalence relation

$$f \sim g :\Leftrightarrow \lambda\Big(\{x : f(x) \neq g(x)\}\Big) = 0$$

of equality almost everywhere. It is a Banach space with the norm

$$\|f\|_p = \left(\int |f|^p(x)\mathrm{d}x\right)^{1/p},$$

where f is any representative in the equivalence class.

For $p = \infty$ we define $\mathscr{L}^p(\mathbb{R}^d)$ as the space of measureable functions for which

$$\|f\|_\infty = \text{ess}{-}\text{sup}|f| := \inf\Big\{t \in \mathbb{R} : \lambda(f^{-1}(t,\infty)) = 0\Big\} \tag{A.6}$$

is finite. The Lebesgue space $L^p(\mathbb{R}^d)$ is the quotient of $\mathscr{L}^p(\mathbb{R}^d)$ by the same equivalence relation.

Proposition A.5 (Hölder's inequality)**.** *Let $1 \leq p, q \leq \infty$ so that $p^{-1} + q^{-1} = 1$, with the convention that $\infty^{-1} = 0$. Then for $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ we have $fg \in L^1(\mathbb{R}^d)$ and*

$$\left| \int f(x)g(x)\mathrm{d}x \right| \leq \|f\|_p \|g\|_q. \tag{A.7}$$

For $d > 1$ an important result concerns the relation of the d-dimensional integral and the iteration of lower-dimensional integrals.

Theorem A.6. *Fubini-Tonelli Let $n, m \geq 1$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ be a measurable function and $A \in \mathscr{B}(\mathbb{R}^{n+m})$.*

a) If $f \geq 0$, then

$$\int_A f(x,y)\lambda(\mathrm{d}(x,y)) = \int_{\pi_1(A)} \Bigg(\int_{\pi_1^{-1}(\{x\})\cap A} f(x,y)\mathrm{d}y \Bigg) \mathrm{d}x = \int_{\pi_2(A)} \Bigg(\int_{\pi_2^{-1}(\{y\})\cap A} f(x,y)\mathrm{d}x \Bigg) \mathrm{d}y$$

where $\pi_j(A)$, $j = 1, 2$ are the projections of A to $\mathbb{R}^n$, $\mathbb{R}^m$ respectively, and the equality is understood in the sense that if one expression is infinite, all are.

b) If f is integrable on A, then

a) The functions

$$x \mapsto f(x,y), \qquad y \mapsto f(x,y)$$

are integrable on $\pi_2^{-1}(\{y\})\cap A$ for almost every $y \in \mathbb{R}^m$, respectively on $\pi_1^{-1}(\{x\})\cap A$ for almost every $x \in \mathbb{R}^n$;

b) the functions (set equal to zero where the integral is not defined)

$$\varphi(y) = \int_{\pi_2^{-1}(\{y\})\cap A} f(x,y)\mathrm{d}x, \qquad \psi(x) = \int_{\pi_1^{-1}(\{x\})\cap A} f(x,y)\mathrm{d}y$$

are integrable;

c) the identity

$$\int_{\pi_2(A)} \varphi(y)\mathrm{d}y = \int_A f(x,y)\mathrm{d}y = \int_{\pi_1(A)} \psi(x)\mathrm{d}x$$

holds.

The well-known transformation formular holds for the Lebesgue integral.

Theorem A.7 (Change of variables)**.** *Let $A \in \mathscr{B}(\mathbb{R}^d)$, let $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be a C^1-diffeomorphism, and denote by $|J(x) := |\det D\varphi(x)|$. Then if f is integrable on A, $x \mapsto f(\varphi(x))|J(x)|$ is integrable on $\varphi^{-1}(A)$ and*

$$\int_A f(x)\mathrm{d}x = \int_{\varphi^{-1}(A)} f(\varphi(x))|J(x)|\mathrm{d}x.$$

The most important properties of the Lebesgue integral are the convergence theorems.

Theorem A.8 (Monotone Convergence). *Let $(f_n)_n \in \mathbb{N}$ be a sequence of measurable functions with $f_n \leq f_{n+1}$ and*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

almost everywhere for some function $f : \mathbb{R}^d \to \mathbb{C}$. Then f is measureable and

$$\lim_{n\to\infty} \int f_n(x)\mathrm{d}x = \int f(x)\mathrm{d}x.$$

Theorem A.9 (Dominated Convergence). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions and assume there is a measurable function f so that*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

almost everywhere. Assume moreover that there exists a positive, integrable function g so that

$$\forall n \in \mathbb{N} : |f_n| \leq g$$

almost everywhere. Then f is integrable and

$$\lim_{n\to\infty} \int f_n(x)\mathrm{d}x = \int f(x)\mathrm{d}x.$$

An important corollary to this result concerns the exchange of integration and differentiation.

Corollary A.10. *Let $U \subset \mathbb{R}^k$ be open and $f : U \times \mathbb{R}^d \to \mathbb{C}$ a measurable function such that*

1. *for all $\eta \in U$, $x \mapsto f(\eta, x)$ is integrable,*
2. *for almost all $x \in \mathbb{R}^d$, $\eta \mapsto f(\eta, x)$ is continuously differentiable,*
3. *there exists a positive, integrable function $g : \mathbb{R}^d \to \mathbb{R}_+$ with*

$$\forall \eta \in U : \big|\nabla_\eta f(\eta, x)\big| \leq g(x).$$

Then $\eta \mapsto \int f(\eta, x)\mathrm{d}x$ is continuously differentiable and for all $j = 1, \dots, k$

$$\partial_{\eta_j} \int f(\eta, x)\mathrm{d}x = \int \partial_{\eta_j} f(\eta, x)\mathrm{d}x.$$

B. Problems

Problem 1. Part 1
For any $n \in \mathbb{N}$, we set $f_n := \mathbf{1}_{[n,n+1]}$.

1. Show that for any $x \in \mathbb{R}_+$, $\lim_{n\to+\infty} f_n(x) = 0$
2. Show that for any $n \in \mathbb{N}$, we have $\int_{\mathbb{R}_+} f_n(x)dx = 1$

Part 2
We will show that the sequence $(f_n)_{n\in\mathbb{N}}$ does not satisfy the following property: there exist a non-negative function $g \in L^1(\mathbb{R}_+)$ such that

$$\text{a.e. } x \in \mathbb{R}_+,\ \forall n \in \mathbb{N}, \quad |f_n(x)| \leq g(x). \tag{B.1}$$

1. Show that for any $x \in \mathbb{R}_+$
$$\sup_{n\in\mathbb{N}}\{|f_n(x)|\} = 1.$$
2. Show that, if a measurable function $g : \mathbb{R}_+ \to \mathbb{R}$ satisfying (B.1), then $g \notin L^1(\mathbb{R}_+)$.

Problem 2. Let $a \in \mathbb{C}$ such that $\mathrm{Re}(a) > 0$. The goal of this exercise is to show that

$$\forall x \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-\frac{|x|^2}{2a}}\,dx = (2a\pi)^{\frac{d}{2}} e^{-a|\xi|^2} \tag{B.2}$$

Part 1
For any $x \in \mathbb{R}$, we define $h(x) := e^{-\frac{x^2}{2a}}$. We assume that $h \in \mathscr{S}(\mathbb{R}^d)$.

1. Show that $h'(x) = -\frac{x}{a}h(x)$.
2. Show that $h^{'} \in L^1(\mathbb{R}^d)$ and that $\widehat{h'}(\xi) = i\xi\widehat{h}(\xi)$.
3. Show that $\widehat{h}'(\xi) = -i\widehat{xh}(\xi)$.
4. Recall that
$$\int_{\mathbb{R}} h(x)dx = \sqrt{2a\pi}.$$
Show that $\widehat{h}(0) = \sqrt{2a\pi}$.
5. Deduce that $\widehat{h}$ is the solution of the following Cauchy problem
$$\begin{cases} \widehat{h}'(\xi) = -ia\xi\widehat{h}(\xi) \text{ in } \mathbb{R}, \\ \widehat{h}(0) = \sqrt{2a\pi}. \end{cases} \tag{B.3}$$

6. Deduce from the Cauchy-Lipschitz theorem that, for any $\xi \in \mathbb{R}$
$$\widehat{h}(\xi) = \sqrt{2\pi a}e^{-a|\xi|}.$$

Part 2 By remarking that for any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have
$$e^{-\frac{|x|^2}{2a}} = \prod_{j=1}^{d} h(x_j),$$
show Formula (B.2).

Problem 3. B.0.1. (The heat equation)

Let $u_0 \in \mathscr{S}(\mathbb{R}^d)$. For any $t \geq 0$ and $\xi \in \mathbb{R}^d$, we set
$$u(t,x) := \frac{1}{(2\pi)^d} \int_{R^d} e^{ix\cdot\xi} e^{-t|\xi|^2} \widehat{u}_0(\xi) d\xi.$$

Part 1

1. Show that for any $(t,x) \in (0,+\infty)\times\mathbb{R}^d$, we have $\partial_t u(t,x) := \int_{\mathbb{R}^d} (-|\xi|^2) e^{ix\cdot\xi} e^{-t|\xi|^2} \widehat{u}_0(\xi) d\xi$.
2. Show that $u \in \mathscr{C}^\infty((0,+\infty) \times \mathbb{R}^d)$.
3. Show that $\partial_t u - \Delta u = 0$ in $(0,+\infty) \times \mathbb{R}^d$.

Part 2

1. Show that
$$\forall (t,x) \in (0,+\infty)\times\mathbb{R}^d, \ \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t|\xi|^2} \widehat{u}_0(\xi) d\xi = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy.$$
2. Show that $\lim_{t\to 0^+} u(t,x) = u_0(x)$.
3. Deduce that for any $x \in \mathbb{R}^d$, we have $u(0,x) = u_0(x)$.

Part 3

Show that, for any $f \in \mathscr{S}(\mathbb{R}^d)$, we have
$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi) d\xi.$$

Problem 4. For multiindeces $\alpha, \beta \in \mathbb{N}^d$, we declare that $\beta \leq \alpha$ if $\beta_j \leq \alpha_j$ for all $j = 1, \ldots, d$. Denote by
$$\binom{\alpha}{\beta} = \prod_{j=1}^{d} \binom{\alpha_j}{\beta_j}.$$

Prove the generalised Leibniz formula for $f, g \in C^{|\alpha|}(\mathbb{R}^d)$
$$\partial^\alpha (fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g).$$

Problem 5. Let $u_0 \in \mathscr{S}(\mathbb{R}^d)$. For any $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ we set

$$u(t,x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{it|\xi|} \widehat{u}_0(\xi) d\xi.$$

a) Show that $u \in \mathscr{C}^\infty(\mathbb{R} \times \mathbb{R}^d)$.

b) Show that u solves the Schrödinger equation

$$\begin{cases} \partial_t u + i\Delta u = 0, & \text{in } \mathbb{R} \times \mathbb{R}^d, \\ \lim_{t\to 0} u(t,x) = u_0(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{B.4}$$

Problem 6. Let u_0 and u_1 in $\mathscr{S}(\mathbb{R}^d)$. For any $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ we set

$$u(t,x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \cos(t|\xi|) \widehat{u}_0(\xi) d\xi + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \frac{\sin(t|\xi|)}{|\xi|} \widehat{u}_1(\xi) d\xi.$$

1. Show that $u \in \mathscr{C}^\infty(\mathbb{R} \times \mathbb{R}^d)$.
2. Show that u solves the wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & \text{in } \mathbb{R} \times \mathbb{R}^d, \\ \lim_{t\to 0} u(t,x) = u_0(x) \text{ and } \lim_{t\to 0} \partial_t u(t,x) = u_1(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{B.5}$$

C. Notation

Symbol	Explanation	Page
$\mathbb{N}$	Natural numbers (not including zero!)	
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$	
D	Differential of a vector-valued function	
grad	Gradient of a scalar function, $\operatorname{grad} f = Df$	
div	Divergence of a vector field, $\operatorname{div} v = \operatorname{Tr}(Dv)$	
$B(x,r)$	Open ball of radius r around x	
$\mathscr{S}(\mathbb{R}^d)$	Space of Schwartz functions on $\mathbb{R}^d$	5
$\mathscr{S}'(\mathbb{R}^d)$	Space of tempered distributions on $\mathbb{R}^d$	??
$L^p(\mathbb{R}^d)$	Lebesgue space of p-integrable functions	11
$H^k(\mathbb{R}^d)$	Sobolev space of functions in $L^2(\mathbb{R}^d)$ with k weak derivatives in L^2	??
X	Usually a complex Banach space	
$\mathrm{B}(X,Y)$	Banach space of bounded linear operators from X to Y	
$\mathrm{B}(X)$	Banach space of bounded linear operators from X to X	
X'	Space of continuous linear functionals on X (=$\mathrm{B}(X,\mathbb{C})$)	??
$\mathcal{H}$	Complex (separable) Hilbert space	
$A, D(A)$	Densely defined linear operator	??
$\mathscr{G}(A)$	Graph of A	??
$\overline{A}$	Closure of $(A, D(A))$	??
$\lVert\cdot\rVert_{D(A)}$	Graph norm on $D(A)$	??
A^*	(Hilbert) adjoint of $(A, D(A))$	??,??
$\ker(A)$	Kernel of A	
$\operatorname{ran}(A)$	Range of A	
$\rho(A)$	Resolvent set of A	??
$R_z(A)$	Resolvent of A in $z \in \rho(A)$, $(A-z)^{-1}$	??
$\sigma(A)$	Spectrum of A	??
$C^k(U)$	Space of k-times continuously differentiable functions $U \to \mathbb{C}$	
$C_0^k(U)$	Space of k-times continuously differentiable functions $U \to \mathbb{C}$ with compact support, $\operatorname{supp} f \Subset U$	

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