# Sheet 8

### Exercise 1.1 (Multiplication operators)

For a (possibly unbounded) measurable function $g : \mathbb{R}^d \to \mathbb{C}$ consider the linear map $M_g$ in $L^2(\mathbb{R}^d)$ defined by

$$\begin{aligned} \mathcal{D}(M_g) &:= \left\{ f \in L^2(\mathbb{R}^d) \,\middle|\, gf \in L^2(\mathbb{R}^d) \right\} \\ (M_g f)(x) &:= g(x)f(x)\,. \end{aligned}$$

Prove:

1. $\mathcal{D}(M_g)$ is dense in $L^2(\mathbb{R}^d)$.
2. $(M_g)^* = M_{\overline{g}}$.
3. $M_g$ is closed.
4. If $g \in L^\infty(\mathbb{R}^d)$ then $M_g$ is bounded, and
$$\|M_g\| = \|g\|_\infty = \sup \Big\{ t : \big|\{x \in \mathbb{R}^d : |g(x)| \geq t\}\big| > 0 \Big\},$$
where $|V|$ denotes the Lebesgue measure of a measurable subset $V \subset \mathbb{R}^d$.
5. $M_g$ is not bounded if $g \notin L^\infty(\mathbb{R}^d)$.

### Exercise 1.2 (Theorem de Cauchy-Lipschitz)

Let $X$ be a Banach space and $F : X \to X$ be a Lipschitz map.

1. Show that for any $x_0 \in X$, there exists $x \in C([0, +\infty[; X) \cap C^1(]0, +\infty[; X)$ such that
$$\begin{cases} x' = F(x), & \text{in } ]0, +\infty[, \\ x(0) = x_0. \end{cases} \tag{1}$$
2. Show that, for a given initial data $x_0 \in X$, this solution is unique.

### Exercise 1.3

Let $A \in \mathrm{B}(\mathbb{C}^d) = \mathbb{C}^{d \times d}$ and consider the linear autonomous ODE

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au(t).$$

Show that

$$\limsup_{t \to \infty} |u(t)| < \infty$$

holds for all solutions if and only if all eigenvalues of $A$ have non-positive real part and the purely imaginary eigenvalues have equal algebraic and geometric multiplicity.

Give examples where the solution exhibits exponential/polynomial growth.

**Exercise 1.4 (Transport equation)**

Let $v \in \mathbb{R}^d$ and $A = v \cdot \nabla$ with $D(A) = H^1(\mathbb{R}^d)$.

1. Show that $A$ is maximal dissipative.
2. Show that for $u_0 \in L^2(\mathbb{R}^d)$

$$(\mathrm{e}^{At} u_0)(x) = u_0(x + tv).$$

3. Determine the spectrum of $T(t) = \mathrm{e}^{tA}$ and its decomposition into $\sigma_p$, $\sigma_c$, $\sigma_r$.

**Exercise 1.5 (Dissipative matrices)**

Let $d \in \mathbb{N}$ and $A \in \mathrm{B}(\mathbb{C}^d)$ be a $d \times d$ matrix.

1. Assume there exists a unitary $U \in \mathrm{B}(\mathbb{C}^d)$ so that $UAU^*$ is diagonal and give a necessary and sufficient condition on $\sigma(A)$ for $A$ to be dissipative.
2. Let $d = 2$ and $A$ be the non-trivial Jordan block

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

   Give a necessary and sufficient condition on $\lambda \in \mathbb{C}$ for $A$ to be dissipative.

3. Let $A$ be as in part $b$) and $\mathrm{Re}\lambda < 0$. Show that there exists a matrix $S$ such that $B = SAS^{-1}$ is dissipative.

**Exercise 1.6 (The wave equation)**

In this exercise we solve the wave equation on $\mathbb{R}^d$ using the Hille Yosida theorem. The wave equation is

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(0) = u_0 \\ \partial_t u(0) = v_0. \end{cases} \tag{W}$$

1. Let $\mathcal{H} := H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ and let $A$ be the operator

$$A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$$

   with domain $D(A) := H^2(\mathbb{R}^d) \oplus H^1(\mathbb{R}^d)$. Show that if $(u, v) \in C^1(\mathbb{R}, \mathcal{H})$ is a solution to the Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(u, v) = A(u, v) \\ (u, v)(0) = (u_0, v_0) \end{cases} \tag{A}$$

   then $u$ solves the wave equation (W).

2. Show that $(u, v)$ solves (A) if and only if $(\tilde{u}, \tilde{v}) = \mathrm{e}^{-t}(u, v)$ solves

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{u}, \tilde{v}) = (A - 1)(\tilde{u}, \tilde{v}) \\ (\tilde{u}, \tilde{v})(0) = (u_0, v_0). \end{cases}$$

3. Show that $A - 1$ is maximal dissipative.

4. State the existence and uniqueness result for the wave equation implied by 1. – 2. and the Hille-Yosida theorem, specifying the functional space for the solution $u$.

**Exercise 1.7 (Ornstein-Uhlenbeck semi-group)**

Let us consider $\gamma : x \in \mathbb{R} \mapsto \frac{1}{\sqrt{2\pi}} e^{-\frac{|x|^2}{2}}$ and set

$$L^2(\gamma) := \left\{ f : \mathbb{R} \mapsto \mathbb{R} \text{ measurable } \middle| \int_{\mathbb{R}} |f(x)|^2 \gamma(x) dx < +\infty \right\}.$$

The for any $t \geq 0$ and $f \in L^2(\gamma)$, we set

$$(U_t f)(x) := \int_{\mathbb{R}} f(e^{-t} x + \sqrt{1 - e^{-2t}} y) \gamma(y) dy.$$

1. Sow that for all $t \geq$ and $f \in L^2(\gamma)$, $U_t f$ is well-define.

2. Show that $(U_t)_{t \geq 0}$ is a contraction semi-group on $L^2(\gamma)$. We denote by $A$ is generator.

3. Let us set

$$\mathcal{A} := \left\{ f \in C^\infty(\mathbb{R}) \middle| \forall \alpha \in \mathbb{N}, \exists P \in \mathbb{R}[X] \text{ such that } |f^{(\alpha)}| \leq P \right\}.$$

   Show that $\mathcal{A} \subset D(A)$ and that for any $f \in \mathcal{A}$, we have

$$\forall x \in \mathbb{R}, \quad (Af)(x) = \partial_x^2 f(x) - x \partial_x f(x).$$

4. We admet that $\mathcal{A}$ is stable by $(\lambda - A)^{-1}$ for some $\lambda > 0$. Show that $\overline{\mathcal{A}}^{D(A)} = D(A)$ (*Hint:* use that $\mathcal{A}$ is dense in $L^2(\gamma)$).

5. Show that $A$ is self adjoint.