Sheet 6

Exercise 1.1 (Weak and strong convergence)

Let $\mathcal{H}$ be a Hilbert space and $(f_n)_{n\in\mathbb{N}}$ a sequence of $\mathcal{H}$ that converges weakly to f in $\mathcal{H}$ and such that $(\|f_n\|_{\mathcal{H}})_{n\in\mathbb{N}}$ converges to $\|f\|_{\mathcal{H}}$. Show that $(f_n)_{n\in\mathbb{N}}$ converges strongly to f in $\mathcal{H}$.

Exercise 1.2 (Density in $H^s(\mathbb{R}^d)$)

Let $s \in \mathbb{R}$. Show that $\mathscr{S}(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$.

Exercise 1.3 (Local compact embedding)

Let $t < s$, $\varphi \in \mathscr{S}(\mathbb{R}^d)$. The goal of this exercise is to show that the multiplication by φ is a compact operator from $H^s(\mathbb{R}^d)$ to $H^t(\mathbb{R}^d)$.

Let $(u_n)_{n\in\mathbb{N}}$ be e sequence of $H^s(\mathbb{R}^d)$ such that $\sup_{n\in\mathbb{N}}\{\|u_n\|_{H^s}\} \leq 1$.

1. Show that, up to extraction of a subsequence, $(u_n)_{n\in\mathbb{N}}$ converging weakly in $H^s(\mathbb{R}^d)$ to an element u.
   Let us set $v_n := u_n - u$.
2. Show that there exists a constant C_1 such that
$$\sup_{n\in\mathbb{N}}\{\|\varphi v_n\|_{H^s}\} \leq C_1.$$
3. Show that for any positive real number R, we have
$$\|\varphi v_n\|_{H^t} \leq \int_{B(0,R)} (1+|\xi|^2)^t |\mathscr{F}(\varphi v_n)|^2 d\xi + \frac{C_1^2}{(1+R^2)^{s-t}}.$$
   Let us consider $\varepsilon > 0$.
4. Show that there exists a positive real number R such that
$$\frac{C_1^2}{(1+R^2)^{s-t}} \leq \varepsilon.$$
5. For all $\xi \in \mathbb{R}^d$, we set $\psi_\xi := \mathscr{F}^{-1}((1+|\cdot|^2)^{-s}\mathscr{F}(\varphi)(\xi - \cdot))$. Show that, for any $\xi \in \mathbb{R}^d$, ψ_ξ belongs to $\mathscr{S}(\mathbb{R}^d)$ and that
$$\forall \xi \in \mathbb{R}^d, \quad \mathscr{F}(\varphi v_n)(\xi) = \langle \psi_\xi, v_n \rangle.$$
6. Deduce that for any $\xi \in \mathbb{R}^d$, we have $\lim_{n\to+\infty} \mathscr{F}(\varphi v_n)(\xi) = 0$.
7. Assume that there exists a positive real number $M > 0$ such that
$$\sup_{\xi\in B(0,R),\, n\in\mathbb{N}} \{|\mathscr{F}(\varphi v_n)|\} \leq M. \tag{1}$$

8. Conclude.
   We will now show (1).

9. Show that there exists a positive real number C_2 such that

$$\forall \mu \in \mathbb{R}^d, \quad |\widehat{\varphi}(\mu)| \leq \frac{C_2}{(1+|\mu|^2)^{\frac{d}{2}+|s|-1}}.$$

10. Show that for any $\xi \in B(0,R)$,

$$\int_{\mathbb{R}^d} (1+|\eta|^2)^{-s} |\widehat{\varphi}(\xi-\eta)|^2 d\eta \\ \leq C_1 \int_{|\eta|\leq 2R} (1+|\eta|^2)^{|s|} ds + C_2 \int_{|\eta|\geq 2R} \frac{(1+|\eta|^2)^{|s|}}{(1+|\xi-\eta|^2)^{\frac{d}{2}+|s|+1}} d\eta.$$

11. Deduce that there exists a positive real number C_3 such that

$$\forall \xi \in B(0,R), \int_{\mathbb{R}^d} (1+|\eta|^2)^{-s} |\widehat{\varphi}(\xi-\eta)|^2 d\eta \leq C_3 (1+R^2)^{|s|+\frac{d}{2}}$$

   (*Hint*: to bound $\int_{|\eta|\geq 2R} (1+|\eta|^2)^{-s}(1+|\xi-\eta|^2)^{-(\frac{d}{2}+|s|+1)} d\eta$, use that if $|\xi| \leq R$ and $|\eta| \geq 2R$, we have $|\xi-\eta| \geq \frac{|\eta|}{2}$.)

12. Deduce that (1) holds.

Exercise 1.4 (Norm of the heat propagator)

Let $t > 0$. Show that $\|e^{t\Delta}\|_{\mathcal{B}(L^2(\mathbb{R}^d))} = 1$.