# Sheet 5

**Exercise 1.1 (The Shrödinger equation and uniqueness)**

For any $t \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^d$, we set

$$k_t(x) := \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{\frac{i|x|^2}{4t}}.$$

1. Let $u_0 \in \mathscr{S}'(\mathbb{R}^d)$. Show that $k_t \star u_0 \in \mathscr{S}'(\mathbb{R}^d))$ and that $t \mapsto k_t \star u_0$ is continuous $C(\mathbb{R}; \mathscr{S}'(\mathbb{R}^d))$. (*Hint*: show that for any $f \in \mathscr{S}(\mathbb{R}^d)$, $t \in \mathbb{R} \mapsto k_t \star f$ belongs to $C(\mathbb{R}; \mathscr{S}'(\mathbb{R}^d))$ and extends this result by duality.)

   In the following, we set $u := k \star u_0$.

2. Show that $u$ satisfies the Schrödinger equation in $\mathscr{S}'(\mathbb{R} \times \mathbb{R}^d)$, that is
$$(i\partial_t + \Delta)u = 0 \text{ in } \mathscr{S}'(\mathbb{R} \times \mathbb{R}^d).$$

3. Show that, for any $\varphi \in \mathscr{S}(\mathbb{R} \times \mathbb{R}^d)$ and $t \in \mathbb{R} \setminus \{0\}$
$$\int_0^t \langle u, (i\partial_t + \Delta)\varphi(s, \cdot)\rangle_{\mathscr{S}'(\mathbb{R}^d), \mathscr{S}(\mathbb{R}^d)} ds$$
$$= \langle u_0, i\varphi(0, \cdot)\rangle_{\mathscr{S}'(\mathbb{R}^d), \mathscr{S}(\mathbb{R}^d)} - \langle u(t), i\varphi(t, \cdot)\rangle_{\mathscr{S}'(\mathbb{R}^d), \mathscr{S}(\mathbb{R}^d)} \quad (1)$$

4. Let $\psi \in \mathscr{S}(\mathbb{R}^d)$, $T > 0$ and $\chi_T \in C_c^\infty(\mathbb{R})$ such that $\chi_T(t) = 1$ for any $t \in [0, T]$. We define the function $\Phi$ by setting for any $(t, \xi) \in \mathbb{R} \times \mathbb{R}^d$, $\Phi^T(t, \xi) := e^{i(T-t)|\xi|^2} \widehat{\psi}(\xi) \chi_T(t)$.
   (a) Show that $\varphi^T : (t, x) \in \mathbb{R} \times \mathbb{R}^d \mapsto \mathscr{F}^{-1}(\Phi^T(t, \xi))$ belongs to $\mathscr{S}(\mathbb{R} \times \mathbb{R}^d)$ and satisfies
$$i\partial_t \varphi^T + \Delta \varphi^T = 0 \text{ in } [0, T] \times \mathbb{R}^d.$$
   (b) Assume that $u_0 = 0$. Deduce from (1) that $u(T) = 0$ in $\mathscr{S}'(\mathbb{R}^d)$.

5. Deduce that for any $u_0 \in \mathscr{S}'(\mathbb{R}^d)$, the distribution $k \star u_0$ is the unique solution of the Schrödinger equation in $\mathscr{S}'(\mathbb{R} \times \mathbb{R}^d)$ satisfying (1) for any $\varphi \in \mathscr{S}(\mathbb{R} \times \mathbb{R}^d)$ and $t \in \mathbb{R} \setminus \{0\}$.

**Exercise 1.2 (Green's function for the Laplacian)**

1. Let $g : \mathbb{R} \to \mathbb{R}$ be the function given by $g(x) = \frac{1}{2}\mathrm{e}^{-|x|}$ for $x \in \mathbb{R}$. Show that $g$ (more precisely the associated distribution, $\varphi_g$) is the unique solution in $\mathscr{S}'(\mathbb{R})$ to the equation
$$(1 - \Delta)\varphi = \delta_0 ,$$
   by

(a) the Fourier transform;

(b) using the distributional derivative.

2. Prove that for $f \in \mathscr{S}(\mathbb{R})$ the unique solution to the equation

$$(1-\Delta)u = f$$

is

$$u(x) = \int g(x-y)f(y)\mathrm{d}y.$$

**Remark 1.2.1.** *The function $g$ is called the fundamental solution or Green's function for the equation.*

### Exercise 1.3 (Example of distributions belonging in $H^s(\mathbb{R}^d)$)

1. Let $a \in \mathbb{R}^d$. Show that $\delta_a \in H^{-s}(\mathbb{R}^d)$ for any $s > d/2$.
2. Let $a, b \in \mathbb{R}$ such that $a < b$. Show that $\mathbf{1}_{[a,b]} \in H^s(\mathbb{R})$ for any $s < \frac{1}{2}$.
3. Why $\mathbf{1}_{[a,b]} \notin H^s(\mathbb{R})$ for $s > \frac{1}{2}$ ?

## Homework (hand in on 12.03.2025).

### Exercise 1.4 (Dispersive estimate for Schrödinger equation)

Let $u_0 \in L^p(\mathbb{R}^d)$ for $p \in [1,2]$ and $u$ be the solution of the Schrödinger equation in the sens of Exercise 1.1, Item 5. with initial data $u_0$.

1. Show that, if $u_0 \in L^1(\mathbb{R}^d)$, then

$$\forall t \in \mathbb{R} \setminus \{0\}, \quad \|u(t)\|_{L^\infty} \leq \frac{1}{(4\pi t)^{\frac{d}{2}}}\|u_0\|_{L^1}$$

2. Show that, if $u_0 \in L^2(\mathbb{R}^d)$, then

$$\forall t \in \mathbb{R} \setminus \{0\}, \quad \|u(t)\|_{L^2} = \|u_0\|_{L^2}.$$

(Hint: use that $\mathscr{F}(k_t) = e^{-it|\cdot|^2}$ in $\mathscr{S}'(\mathbb{R}^d)$.)

3. (Bonus) Let $p'$ a real number such that $\frac{1}{p} + \frac{1}{p'} = 1$. Show that,

$$\forall t \in \mathbb{R} \setminus \{0\}, \quad \|u(t)\|_{L^{p'}} \leq \frac{1}{(4\pi|t|)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{2})}}\|u_0\|_{L^p}.$$

(*Hint*: use the Riesz-Thorin Theorem.)

**Remark 1.4.1.** *The estimate shown in the last question is called a dispersive estimate. These estimates are the main tools to derive the Strichartz estimate for the Schrödinger equation. These estimates are used to solve a large class of nonlinear Schrödinger equations.*

**Exercise 1.5 (Nonlinear heat equation)**

Let $u_0 \in L^d(\mathbb{R}^d)$.

**Part 1: Functional setting.** We define the set

$$\mathrm{K} := \Big\{ u \in \mathscr{C}((0,+\infty); L^{2d}(\mathbb{R}^d)) \Big| \|u\|_K := \sup_{s>0} \{ s^{\frac{1}{4}} \|u(s,\cdot)\|_{L^{2d}} \} < \infty \Big\}.$$

1. Show that K is a Banach space.

**Part 2: Nonlinear estimates.** For any $u$, $v$ and $w$ in K, we define

$$\forall t \geq 0, \quad \mathscr{T}(u,v,w)(t) := \int_0^t e^{(t-s)\Delta} \left( u(s,\cdot)v(s,\cdot)w(s,\cdot) \right) ds.$$

1. In this question we will show that for any $u$, $v$ and $w$ in K, we have $\mathscr{T}(u,v,w) \in \mathrm{K}$ and that there is a positive constant $C$ such that

$$\|\mathscr{T}(u,v,w)\|_{\mathrm{K}} \leq C_1 \|u\|_{\mathrm{K}} \|v\|_{\mathrm{K}} \|w\|_{\mathrm{K}}. \tag{2}$$

   Let $u$, $v$ and $w$ in K.

   (a) Show that

$$\forall t \geq 0, \ \|\mathscr{T}(u,v,w)(t)\|_{L^{2d}} \leq \left( \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} ds \right) \|u(s,\cdot)v(s,\cdot)w(s,\cdot)\|_{L^{2d/3}}.$$

   (b) Deduce that

$$\forall t \geq 0, \quad \|\mathscr{T}(u,v,w)(t)\|_{L^{2d}} \leq \left( \int_0^t \frac{1}{(t-s)^{\frac{1}{2}} s^{\frac{3}{4}}} ds \right) \|u\|_{\mathrm{K}} \|v\|_{\mathrm{K}} \|w\|_{\mathrm{K}}.$$

   (c) Deduce that (2) holds.

   (d) Show that $\mathscr{T}(u,v,w) \in \mathrm{K}$.

**Part 3: Duhamel formula.** Let $u_0 \in L^d(\mathbb{R}^d)$. For any $t \geq 0$ and $u \in \mathrm{K}$, we define

$$\Phi(u)(t) := e^{t\Delta} u_0 + \mathscr{T}(u,u,u)(t).$$

1. Show that there exists a constant $C_2$ such that for any $u \in \mathrm{K}$, we have

$$\|\Phi(u)\|_{\mathrm{K}} \leq C_2 \left( \|u_0\|_{L^d} + \|u\|_{\mathrm{K}}^3 \right) \tag{3}$$

   and that $\Phi \in \mathrm{K}$.

2. Show that there exists a constant $C_3$ such that for any $u$ and $v$ in K, we have

$$\|\Phi(u) - \Phi(v)\|_{\mathrm{K}} \leq C_3 \left( \|u\|_{\mathrm{K}}^2 + \|v\|_{\mathrm{K}}^2 + \|u\|_{\mathrm{K}} \|v\|_{\mathrm{K}} \right) \|u - v\|_{\mathrm{K}}. \tag{4}$$

   (*Hint*: use Estimate (2))

**Part 4: Fixed point argument** Let $\varepsilon > 0$. Assume that

$$\|u_0\|_{L^d} < \varepsilon. \tag{5}$$

and we introduce the set

$$\mathrm{B}(2\varepsilon) := \big\{u \in \mathrm{K} \big| \|u\|_{\mathrm{K}} < 2\varepsilon\big\}.$$

1. Show that $\mathrm{B}(2\varepsilon)$ is a complete metric space for the distance induce be K norm, namely $\|\cdot\|_{\mathrm{K}}$.

2. Show that there exists a constant $C_4$ which does not depends of $\varepsilon$, such that for any $u$ and $v$ in $\mathrm{B}(2\varepsilon)$, we have

$$\|\Phi(u)\|_{\mathrm{K}} \leq C_4(1+\varepsilon^2)\varepsilon$$

   and

$$\|\Phi(u) - \Phi(v)\|_{\mathrm{K}} \leq C_4\varepsilon^2\|u - v\|_{\mathrm{K}}.$$

3. Choose $\varepsilon > 0$ small enough such that $\Phi$ is a strict contraction of $\mathrm{B}(2\varepsilon)$.

4. Deduce that $\Phi$ has a unique fixed point $u$. (*Hint*: use the Banach fixed point theorem.)

**Remark 1.5.1.** *The fixed points of $\Phi$ are called the mild solutions of the equation*

$$\begin{cases} \partial_t u - \Delta u + u^3 = 0, & \textit{in } ]0, +\infty[\times\mathbb{R}^d, \\ u(0, \cdot) = u_0, & \textit{in } \mathbb{R}^d, \end{cases} \tag{6}$$

*for small enough initial data $u_0$. We can obtain the existence of mild solutions for $u_0$ large, but only for small times. The same method (so called "Kato method") can be used to show the existence of solutions to the Navier-Stokes equation.*