Sheet 3

Exercise 1.1 (Heat equation in L^p (I))

For any $t > 0$ and $f \in \mathscr{S}(\mathbb{R}^d)$, we define the function $e^{t\Delta}f$ by

$$e^{t\Delta}f := f \star h_t,$$

where

$$\forall y \in \mathbb{R}^d, \quad h_t(y) := \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|y|^2}{4t}}.$$

1. Let $f \in \mathscr{S}(\mathbb{R}^d)$. What is the Cauchy problem satisfied by $u : (t,x) \in]0,+\infty[\times\mathbb{R}^d \mapsto e^{t\Delta}f \in \mathbb{R}$.
2. Let $p \in [1,\infty[$ and $t > 0$. Show that for any $q \in [p,\infty[$, $e^{t\Delta}$ extends to a continuous operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ and that
$$\|e^{t\Delta}\|_{\mathrm{B}(L^p,L^q)} \leq \|h_t\|_{L^{(1+1/q-1/p)^{-1}}}.$$
3. Show that for any $p \in [1,\infty[$, $q \in [p,\infty[$ and $t > 0$, we have
$$\|e^{t\Delta}\|_{\mathrm{B}(L^p,L^q)} \leq \frac{1}{t^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}}.$$
4. Show that for any $f \in L^p(\mathbb{R}^d)$ with $p \in [1,\infty[$, the function $u : (t,x) \in]0,+\infty[\times\mathbb{R}^d \mapsto e^{t\Delta}f \in \mathbb{R}$ belong in $C^\infty(]0,+\infty[\times\mathbb{R}^d)$ and satisfies the heat equation.
5. Let $p \in [1,\infty[$ and $f \in L^p(\mathbb{R}^d)$. Show that $\lim_{t\to 0^+} e^{t\Delta}f = f$ in $L^p(\mathbb{R}^d)$.

Exercise 1.2 (Schrödinger equation in L^2 (I))

For any $t \in R$ and $f \in \mathscr{S}(\mathbb{R}^d)$, we define the function

$$e^{it\Delta}f := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{it|\xi|^2} \widehat{f}(\xi) d\xi.$$

1. Show that for any $t \in \mathbb{R}$ the operator $e^{it\Delta}$ extends to an operator from $L^2(\mathbb{R}^d)$ into itself and that
$$\forall f \in L^2(\mathbb{R}^d), \quad \|e^{it\Delta}f\|_{L^2} = \|f\|_{L^2}.$$

Homework (hand in on 19.02.2025).

Exercise 1.3 (Schrödinger equation in L^2 (II))

Let t and s in $\mathbb{R}$. Show that

- $e^{i0\Delta} = \mathrm{Id}_{L^2}$.
- $e^{it\Delta} \circ e^{is\Delta} = e^{i(s+t)\Delta}$.
- $(e^{it\Delta})^* = e^{-it\Delta}$.